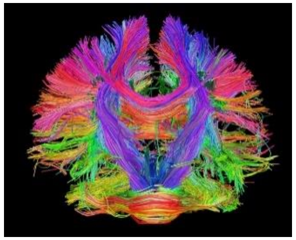



CS-570
Statistical Signal Processing

Lecture 10: Quantization & sampling

Spring Semester 2019

Grigorios Tsagkatakis

Matrix Completion (MC)

Relaxation

$$\min\{\ \|\mathbf{M}\|_* \ :\ \ \mathcal{A}(\mathbf{X}) = \mathcal{A}(\mathbf{M})\}$$

Performance

$$\left\|M - M^*\right\|_F^2 \le 4\sqrt{\frac{(2+p)\min(n_1, n_2)}{p}}\delta + 2\delta,$$

$$\text{where } p = \text{fraction of known entries} = \frac{m}{n_1 n_2} = \frac{|\Omega|}{n_1 n_2}$$

Noisy case

$$\min\{\ \|\mathbf{M}\|_* \ :\ \ \|\mathcal{A}(\mathbf{X}) - \mathcal{A}(\mathbf{M})\|_F^2 \le \epsilon\}$$

CS and MC

	Sparse recovery	*Rank minimization*
Unknown	Vector *x*	Matrix *A*
Observations	$y = Ax$	$y = L[A]$ (linear map)
Combinatorial objective	$\#\{\mathbf{x}_i \neq 0\} \quad = \quad \|\mathbf{x}\|_0$	$\mathrm{rank}(A) \quad = \quad \#\{\sigma_i(A) \neq 0\}$ $= \quad \|\sigma(A)\|_0$
Convex relaxation	$\|\mathbf{x}\|_1 = \sum_i \lvert\mathbf{x}_i\rvert$	$\|A\|_* = \sum_i \sigma_i(A)$
Algorithmic tools	Linear programming	Semidefinite programming

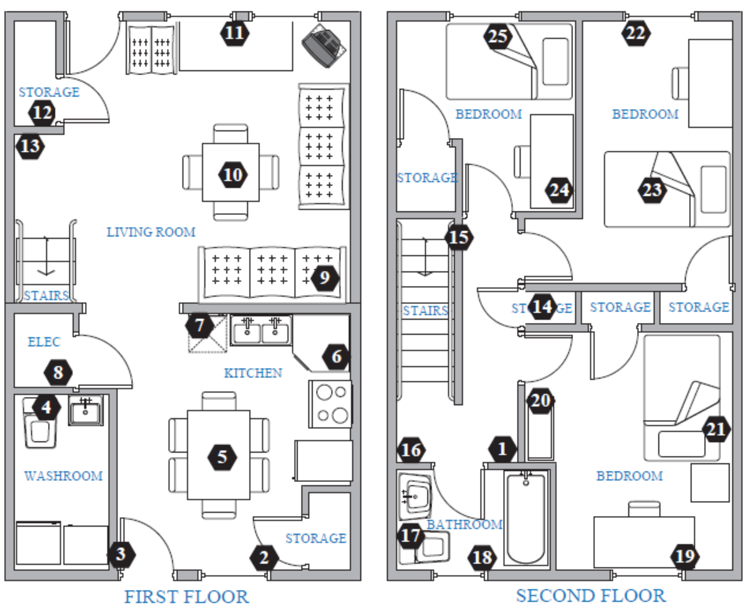
Yi Ma et al, "Matrix Extensions to Sparse Recovery", CVPR2009

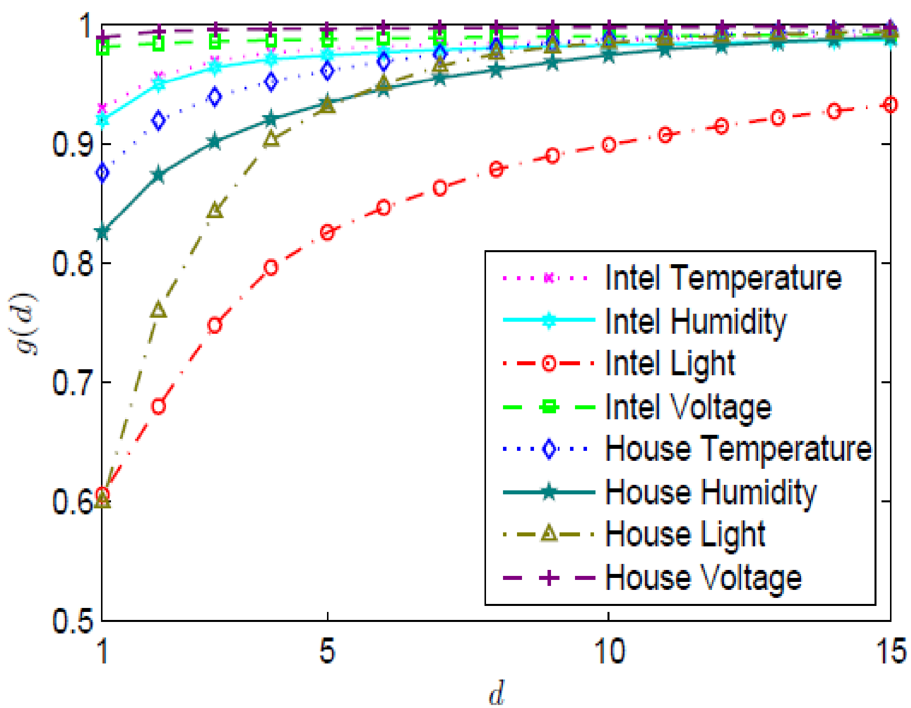
Applications of MC

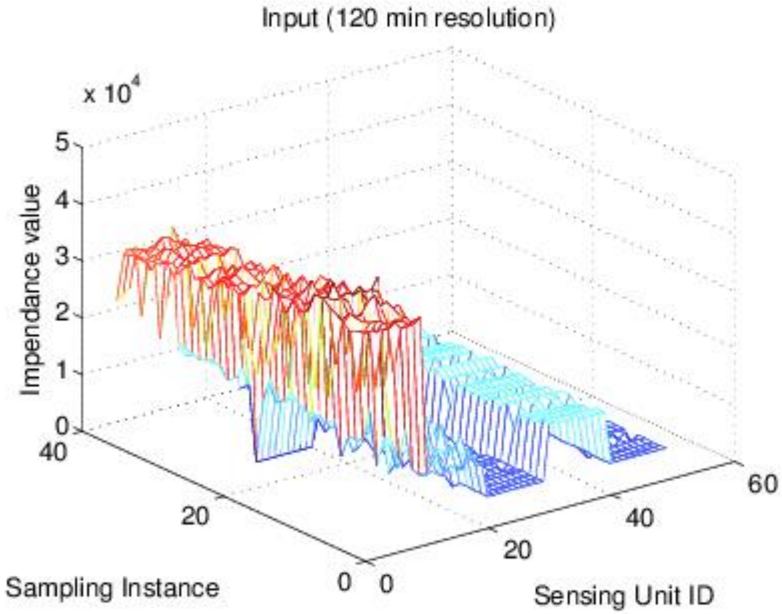
- Recommendation systems
  - Matrix (user, preference/quality/intention)
- Sensor localization
  - Matrix (location, physical quantity)
- Data recovery in Wireless Sensor Networks
  - Matrix (sensor, time)

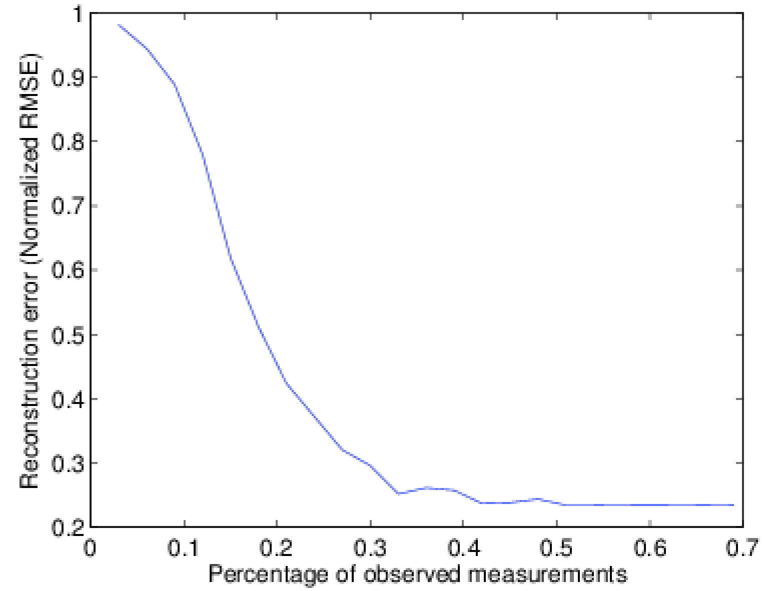
Data Gathering

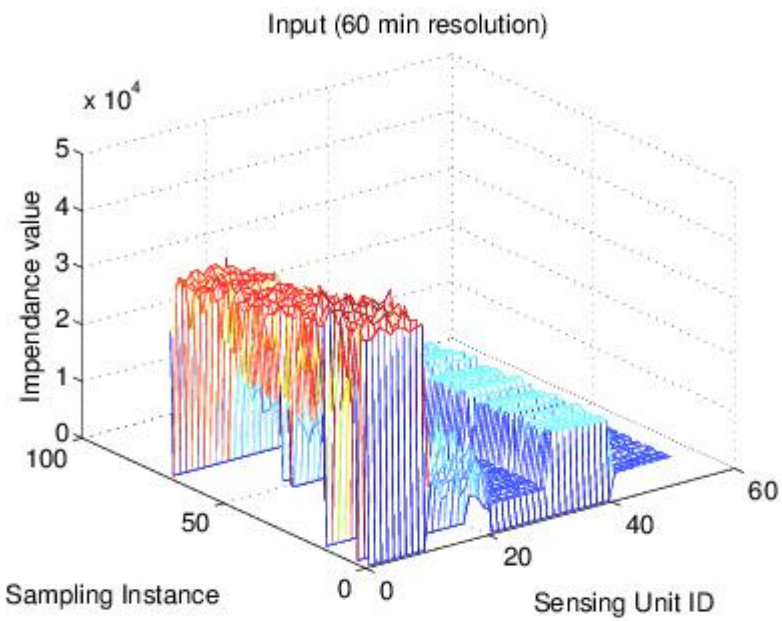
- STCDG: An Efficient Data Gathering Algorithm Based on Matrix Completion for Wireless Sensor Networks

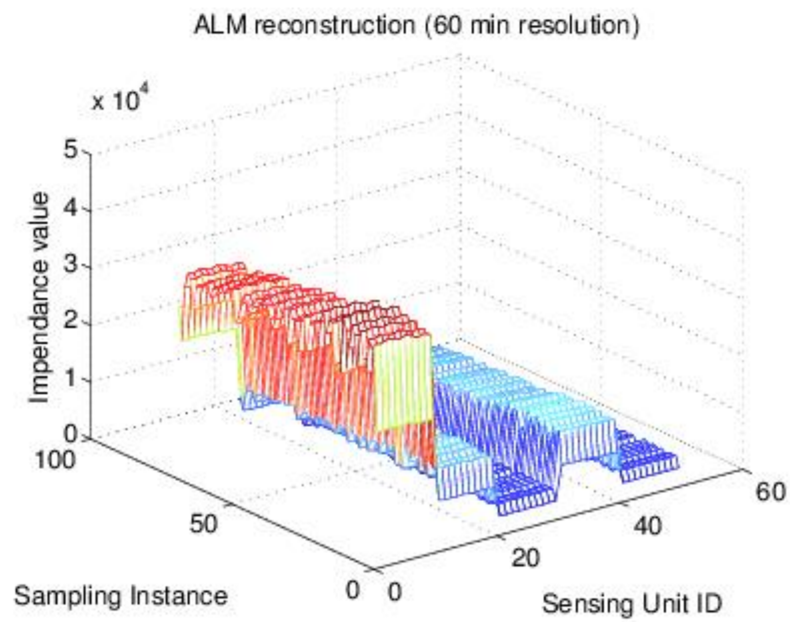




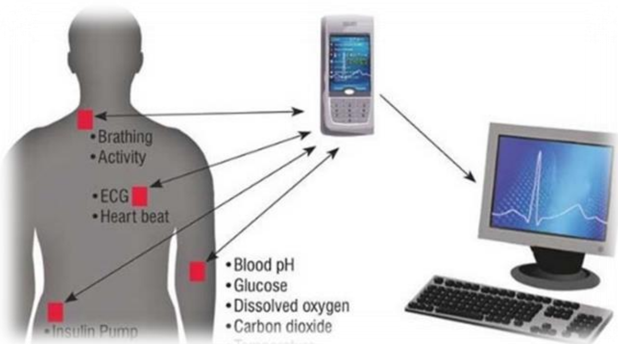

Input (120 min resolution)
x 10^4
Impendance value
Sampling Instance
Sensing Unit ID


Reconstruction error (Normalized RMSE)
Percentage of observed measurements


Input (60 min resolution)
x 10^4
Impendance value
Sampling Instance
Sensing Unit ID


ALM reconstruction (60 min resolution)
x 10^4
Impendance value
Sampling Instance
Sensing Unit ID

WSNs for Human Activity Recognition



Elderly assistance

Chronic conditions management

HAR

Health monitoring

Acting upon alerting events

Fitness coaching

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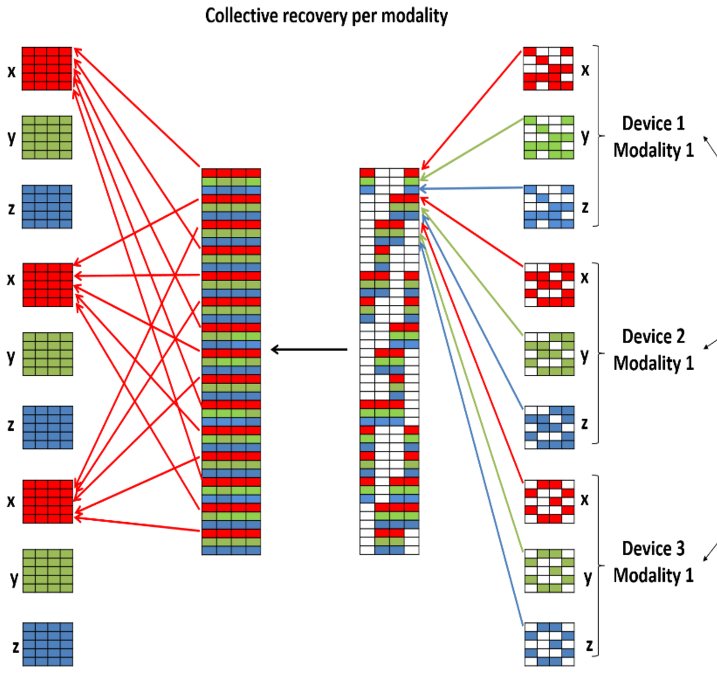
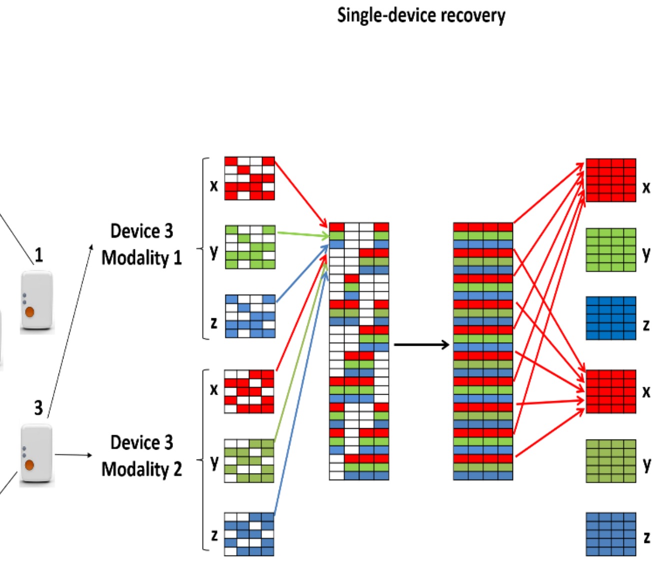
Single-device vs collective recovery: matrices

Scenario 2
Collective per modality

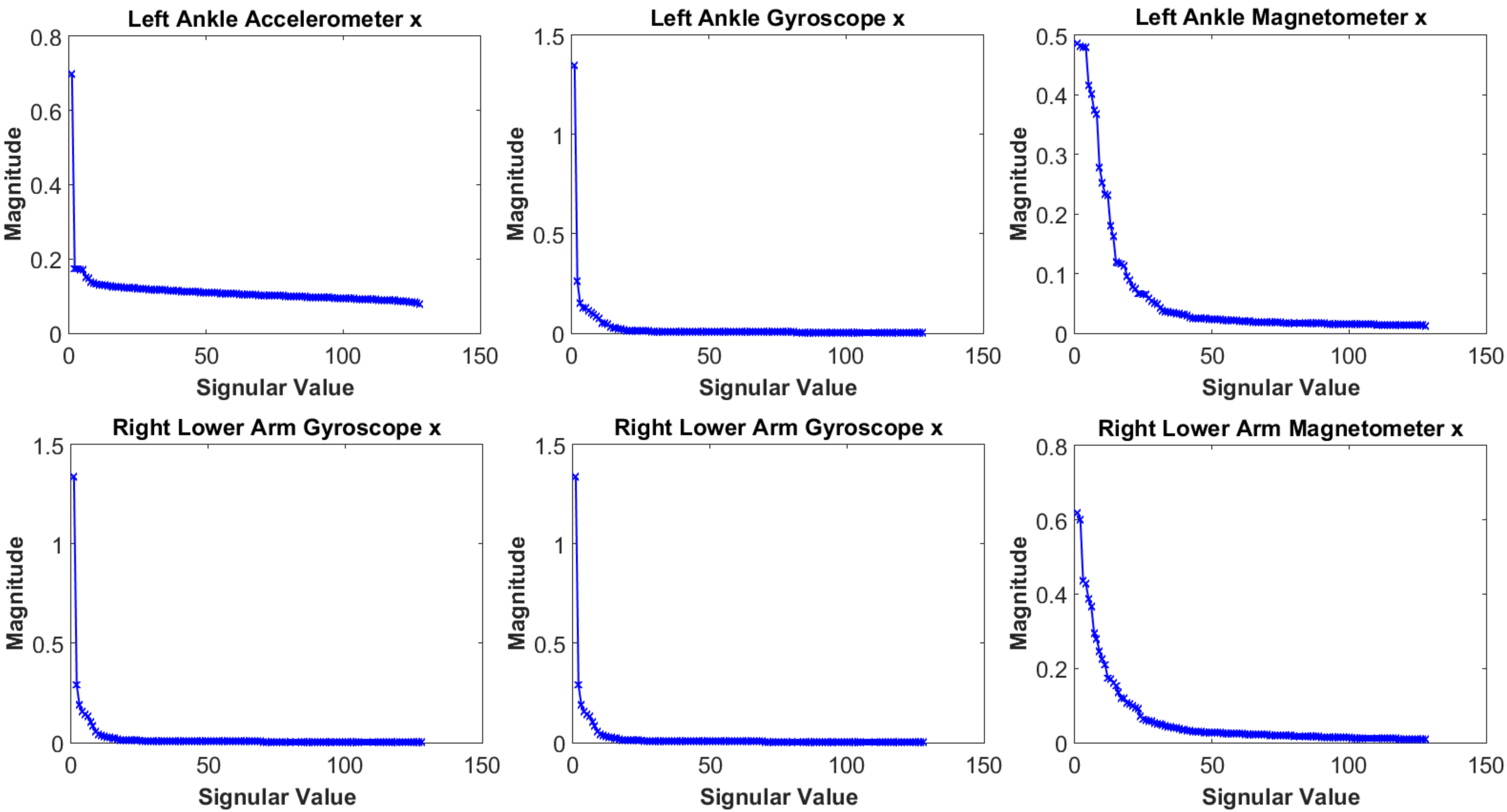
Scenario 1
Single-device





Scenario 3: Overall collective recovery structured similarly

Body sensor network



RTT estimation

Decentralized Matrix Factorization by Stochastic Gradient Descent (DMFSGD),

Estimation of end-to-end network distances

- Network nodes exchange messages with each other
- Each node collects and processes local measurements

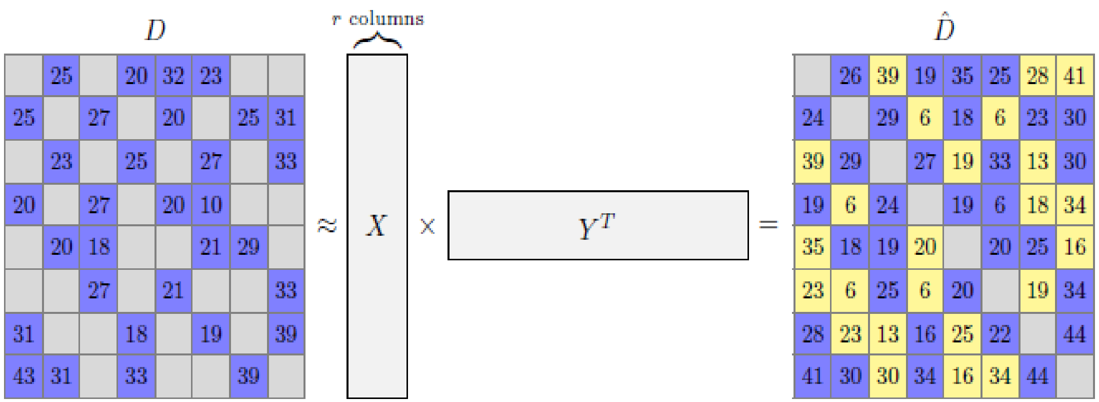


Fig. 2. Network distance prediction by matrix factorization. Note that the diagonal entries of D and $\hat{D}$ are empty.

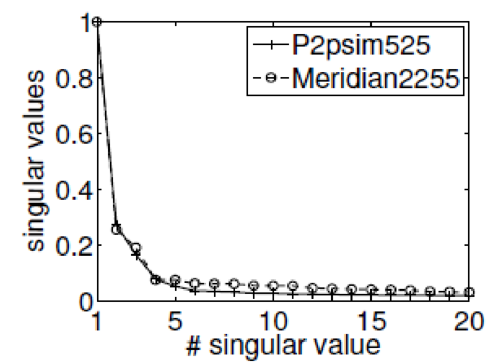
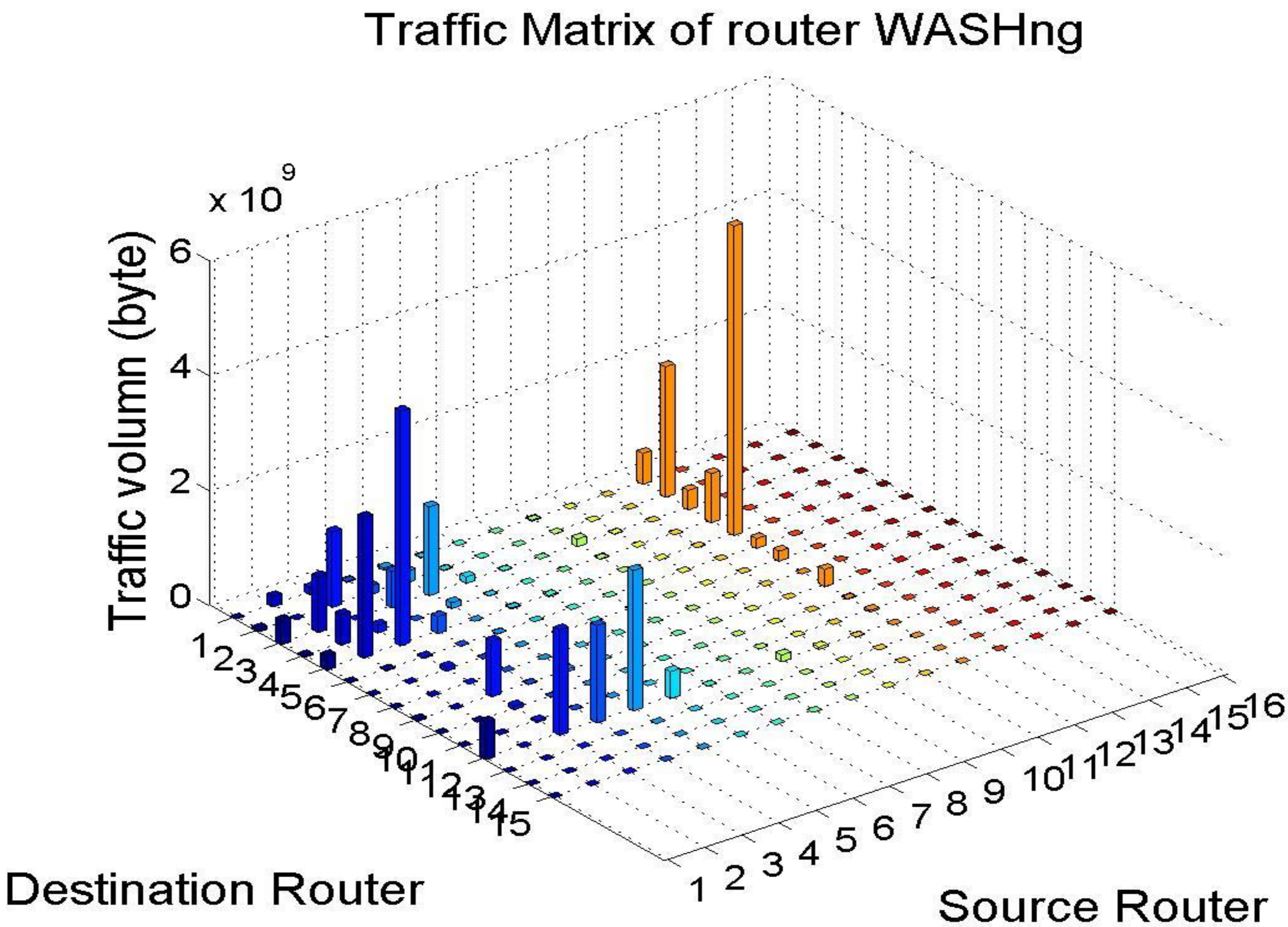
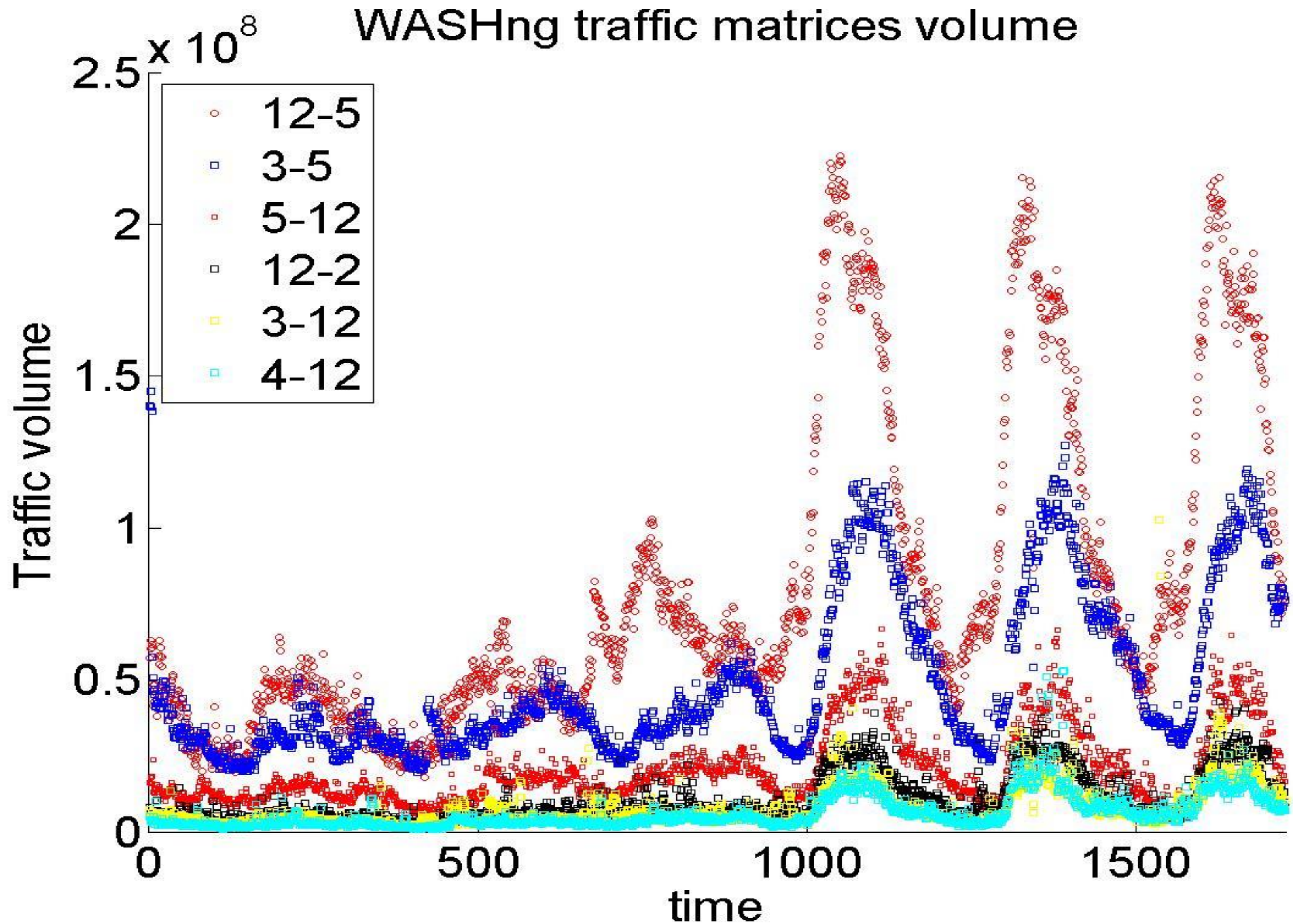
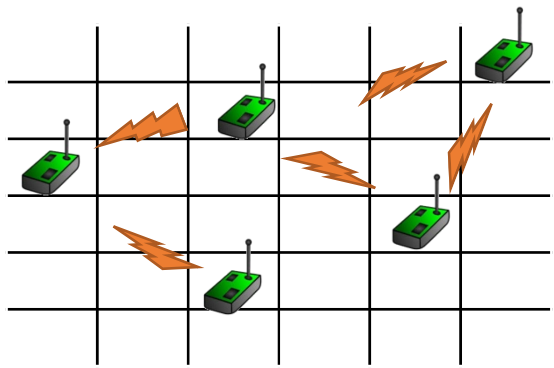


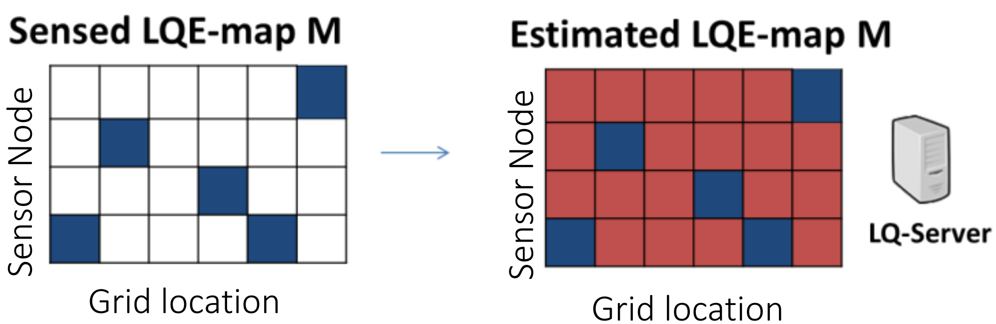
Fig. 3. The singular values of a RTT matrix of 2255×2255, extracted from the Meridian dataset [30] and called "Meridian2255", and of a RTT matrix of 525×525, extracted from the P2psim dataset [30] and called "P2psim525". The singular values are normalized so that the largest singular values of both matrices are equal to 1.


Traffic Matrix of router WASHng
x 10^9
Traffic volumn (byte)
0
2
4
6
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
Destination Router
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Source Router

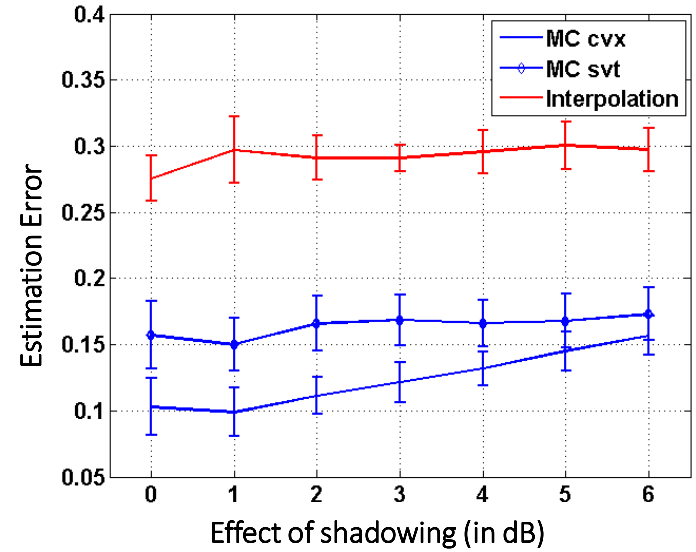

WASHng traffic matrices volume
x 10^8
2.5
2
1.5
1
0.5
0
12-5
3-5
5-12
12-2
3-12
4-12
Traffic volume
0
500
1000
1500
time

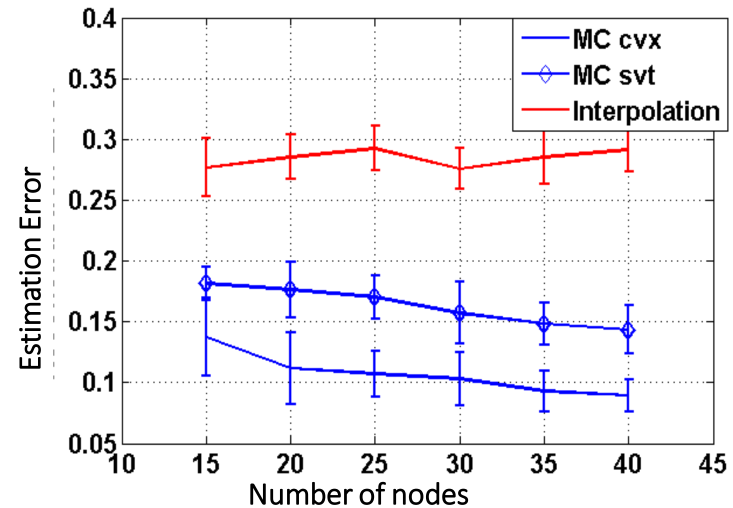
LQM Estimation



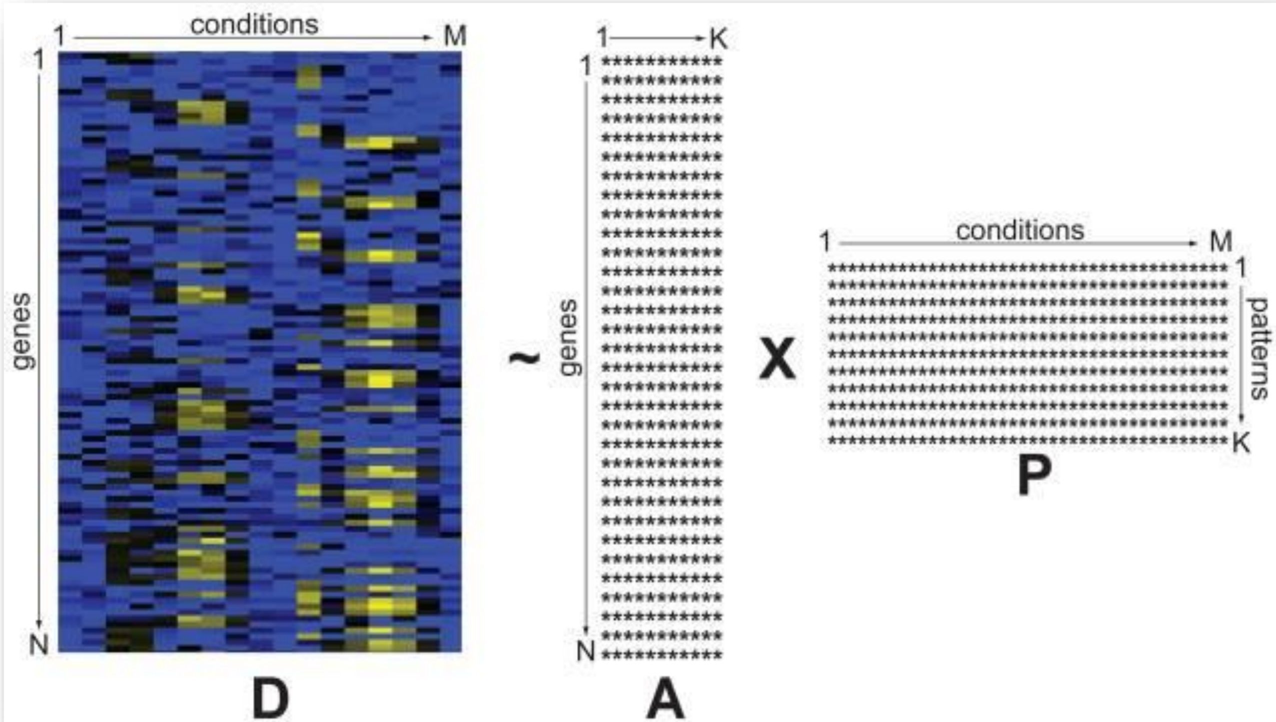


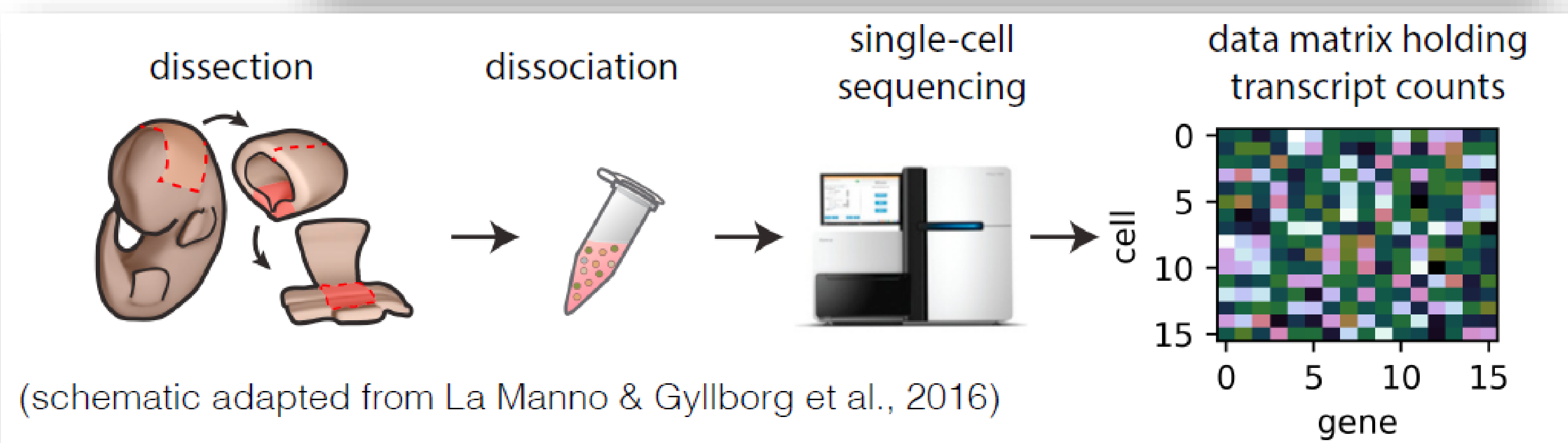
- Dataset
  - Testbed @ FORTH (144m^2, 1x1m grid)
  - RSSI values (channel quality)
  - 13 IEEE802.11b/g channels





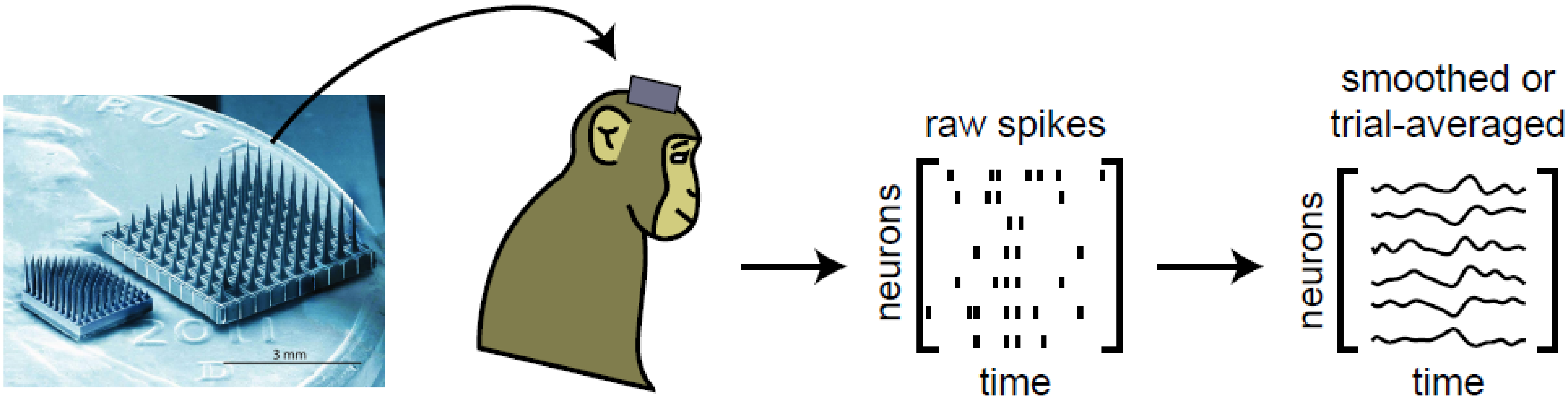
Microarray data

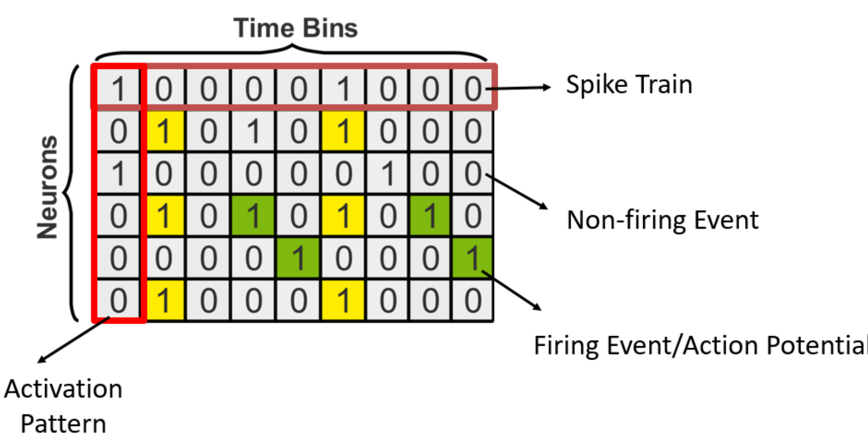


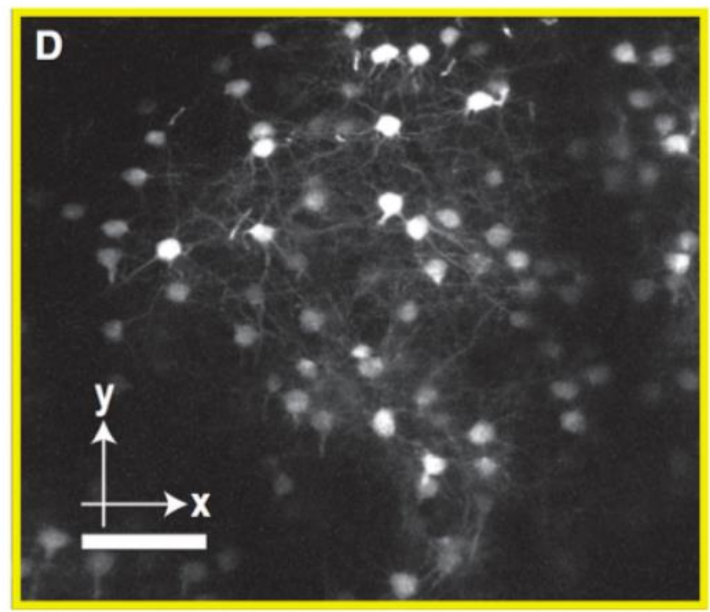


(schematic adapted from La Manno & Gyllborg et al., 2016)

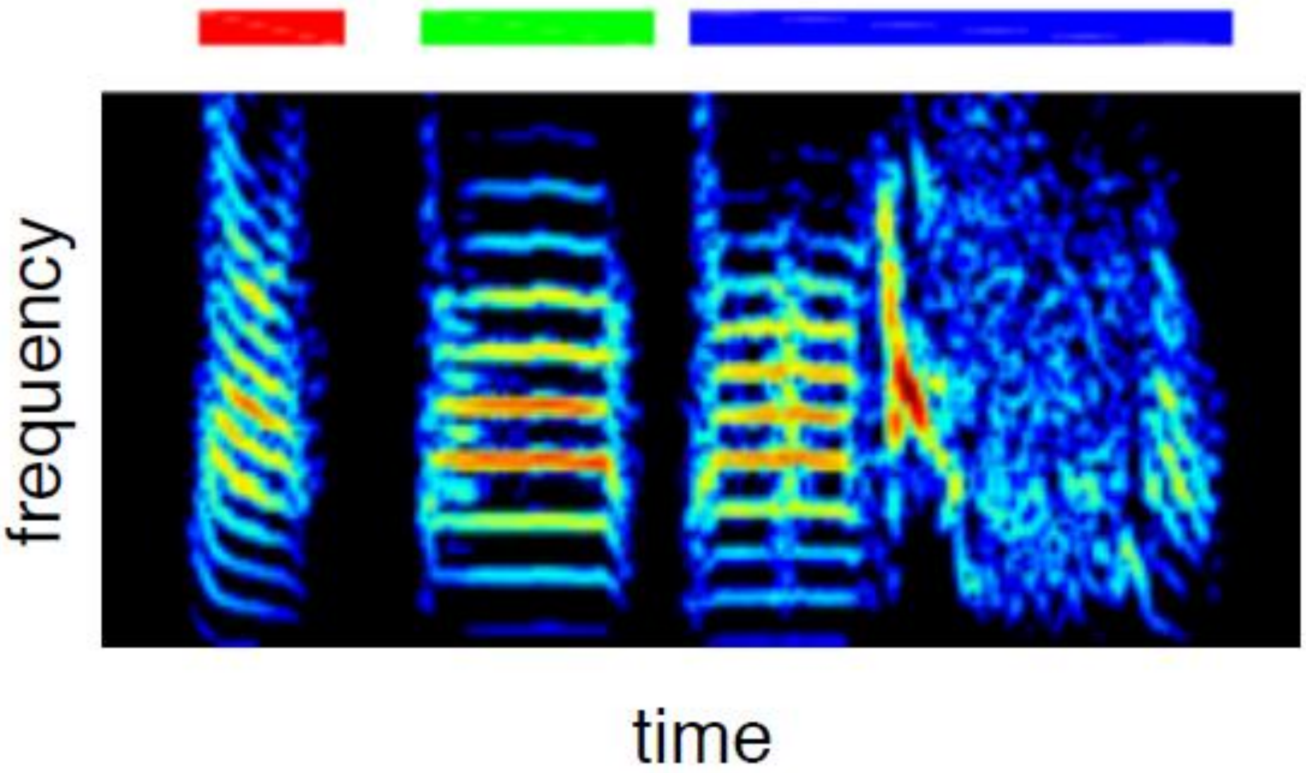
Neural Activity







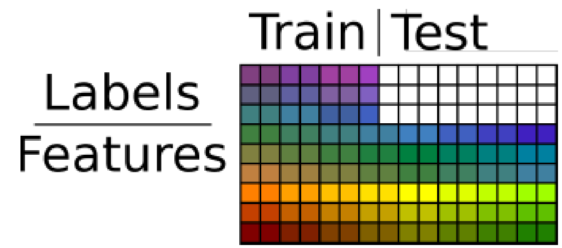
Audio



Transductive Classification

- Unknown labels and corrupted or missing training features and corresponding labels leading to incomplete matrix

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Y}_{tr} & \mathbf{Y}_{tst} \\ \mathbf{X}_{tr} & \mathbf{X}_{tst} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{Y}_{tr} & \mathbf{?} \\ \mathbf{X}_{tr} & \mathbf{X}_{tst} \end{bmatrix}$$



- Formulate classification as rank minimization

$$\begin{aligned} \min \quad & \mu \|\mathbf{Z}\|_* + l_X(\mathbf{Z_X}) + \lambda\, l_Y(\mathbf{Z_Y}) \\ \text{s.t.} \quad & \mathbf{Z_X} \geq \mathbf{0} \ \text{ and } \ \mathbf{Z_1} = \mathbf{1}^{\mathrm{T}} \end{aligned}$$

where:

$$l_X(\mathbf{Z_X}) = \sum_{ij \in \mathbf{Z_X}} (z_{ij} - z_{0ij})^2$$

$$l_Y(\mathbf{Z_Y}) = \sum_{ij \in \mathbf{Y}_{train}} \frac{1}{\gamma} log(1 + exp(-\gamma z_{ij} z_{0ij}))$$

Multi-label image classification

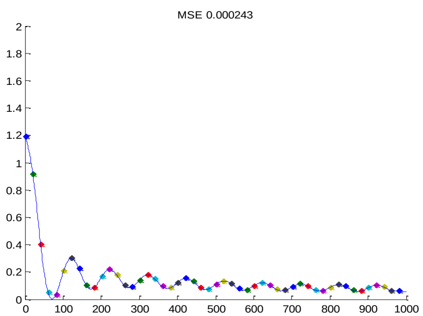
$$\mathbf{Z}^{obs} = \mathbf{Z} + \mathbf{E} = \left[\begin{array}{ccccc|ccc|cccc} 0 & 0 & 1 & \dots & 0 & ? & \dots & ? & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & ? & \dots & ? & 0 & 1 & 0 & 0 \\ 1 & ? & 1 & \dots & 0 & ? & \dots & ? & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \dots & 0 & ? & \dots & ? & 0 & 0 & 0 & 1 \\ \mathbf{h}_1^{tr} & \mathbf{h}_2^{tr} & \mathbf{h}_3^{tr} & \dots & \mathbf{h}_N^{tr} & \mathbf{h}_1^{tst} & \dots & \mathbf{h}_M^{tst} & ? & ? & ? & ? \end{array}\right]$$

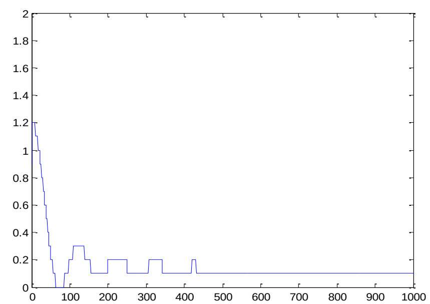
(a) Training (b) Testing (c) Class Representatives

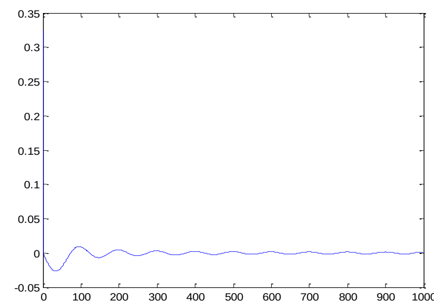
Analog-to-digital transformation

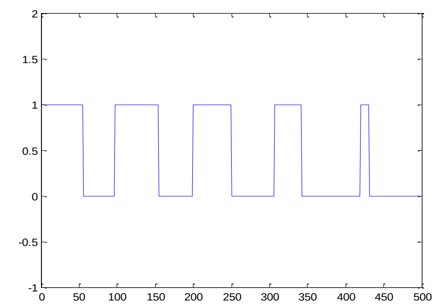
Signal Sensing

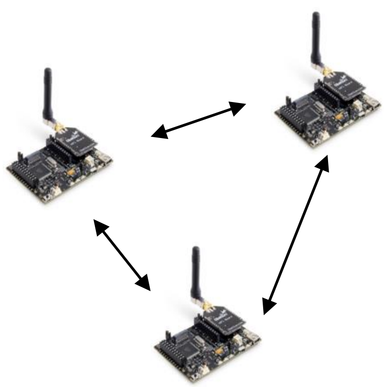
Sample	Quantize	Process	Transmit









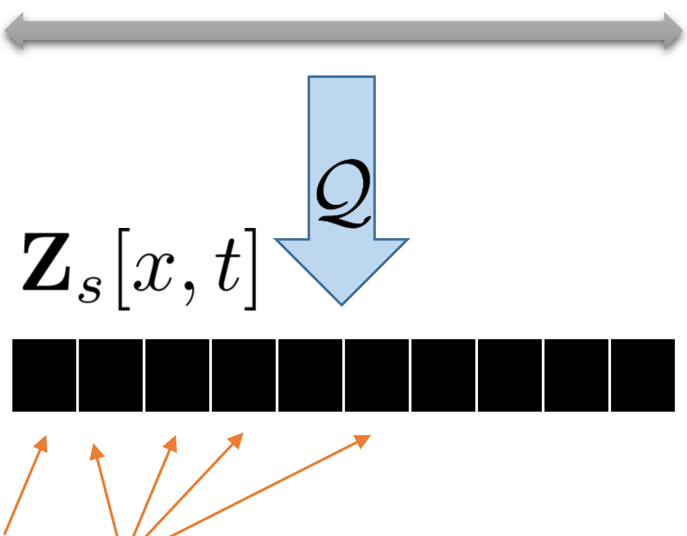


Signal Quantization

Map values to symbols

- Bits per measurements

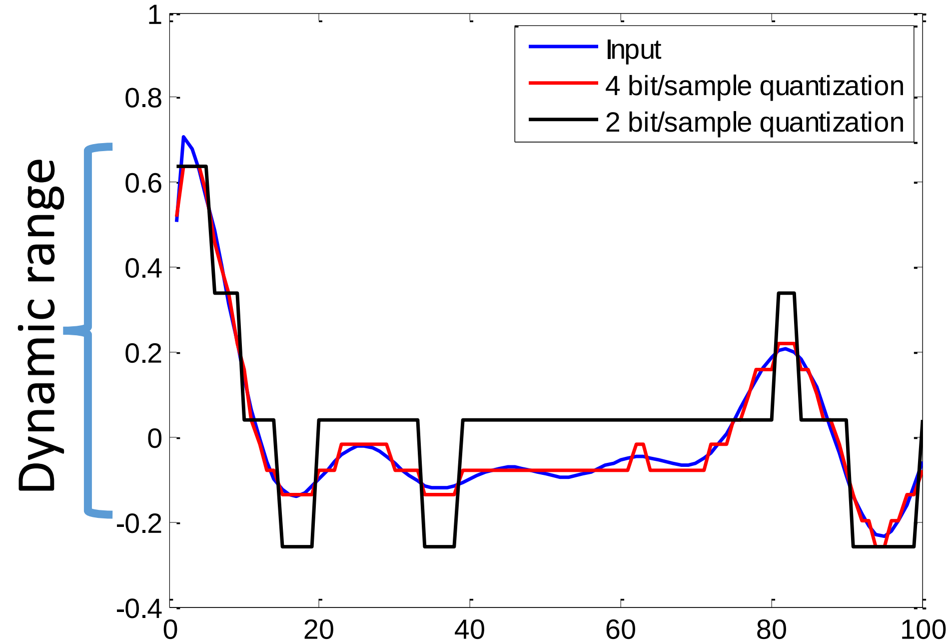
$$\mathbf{Y}_s[x,t]$$



$$\mathcal{Q}$$

$$\mathbf{Z}_s[x,t]$$

$$|\mathcal{C}| = \mathcal{B} = 2^R$$



$$\mathcal{T} = \{t_1, ..t_{\mathcal{B}} | t_i < t_j, \forall i, j\}$$

$$Q(x) = \Delta \cdot \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor = \Delta \cdot \text{floor}\left(\frac{x}{\Delta} + \frac{1}{2}\right)$$

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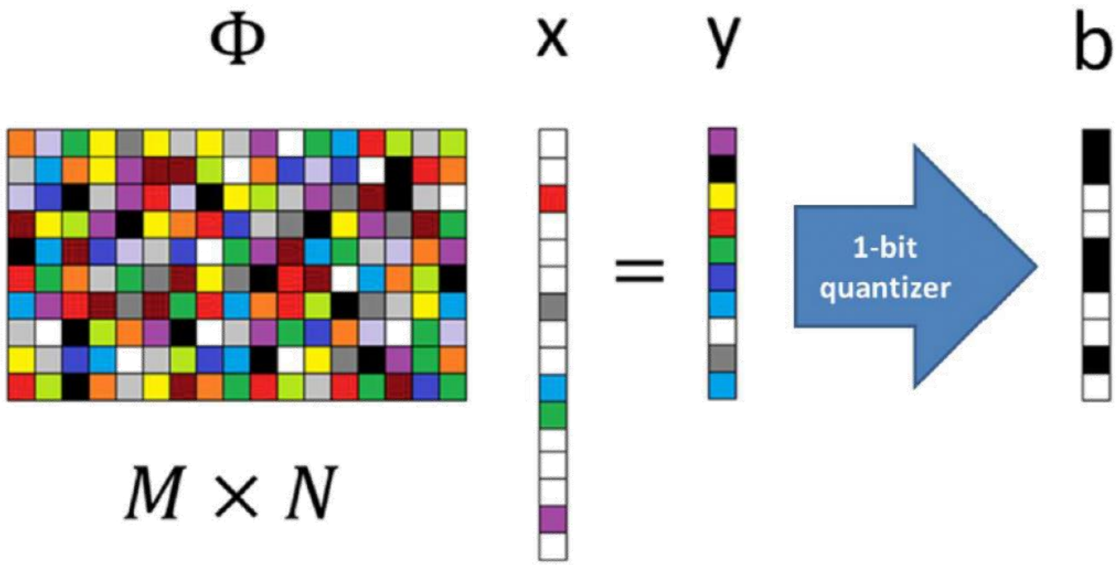
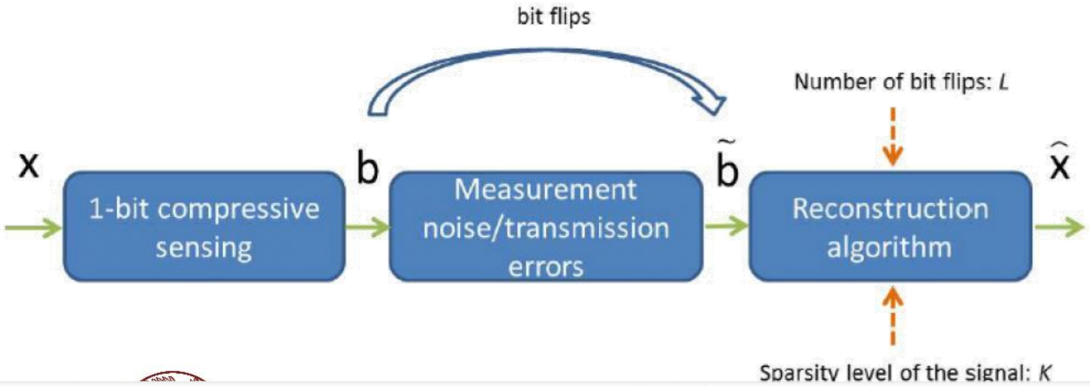
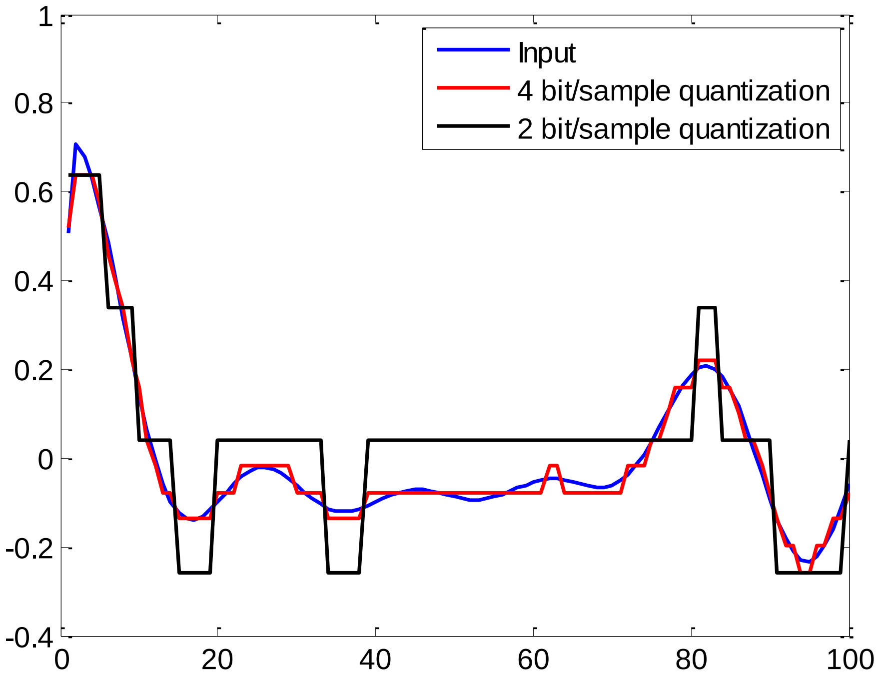
1-Bit Compressive Sensing





End-to-End CS imaging

- Difference aspects of sensing process
  - Dictionary
  - Noise sources
  - **Quantization**
    - ADC
    - Storage
    - Transmission
  - Entropy Coding



(Scalar) Signal Quantization

Map from real to indexed $\mathcal{Q} : \mathbb{R} \rightarrow 2^R$

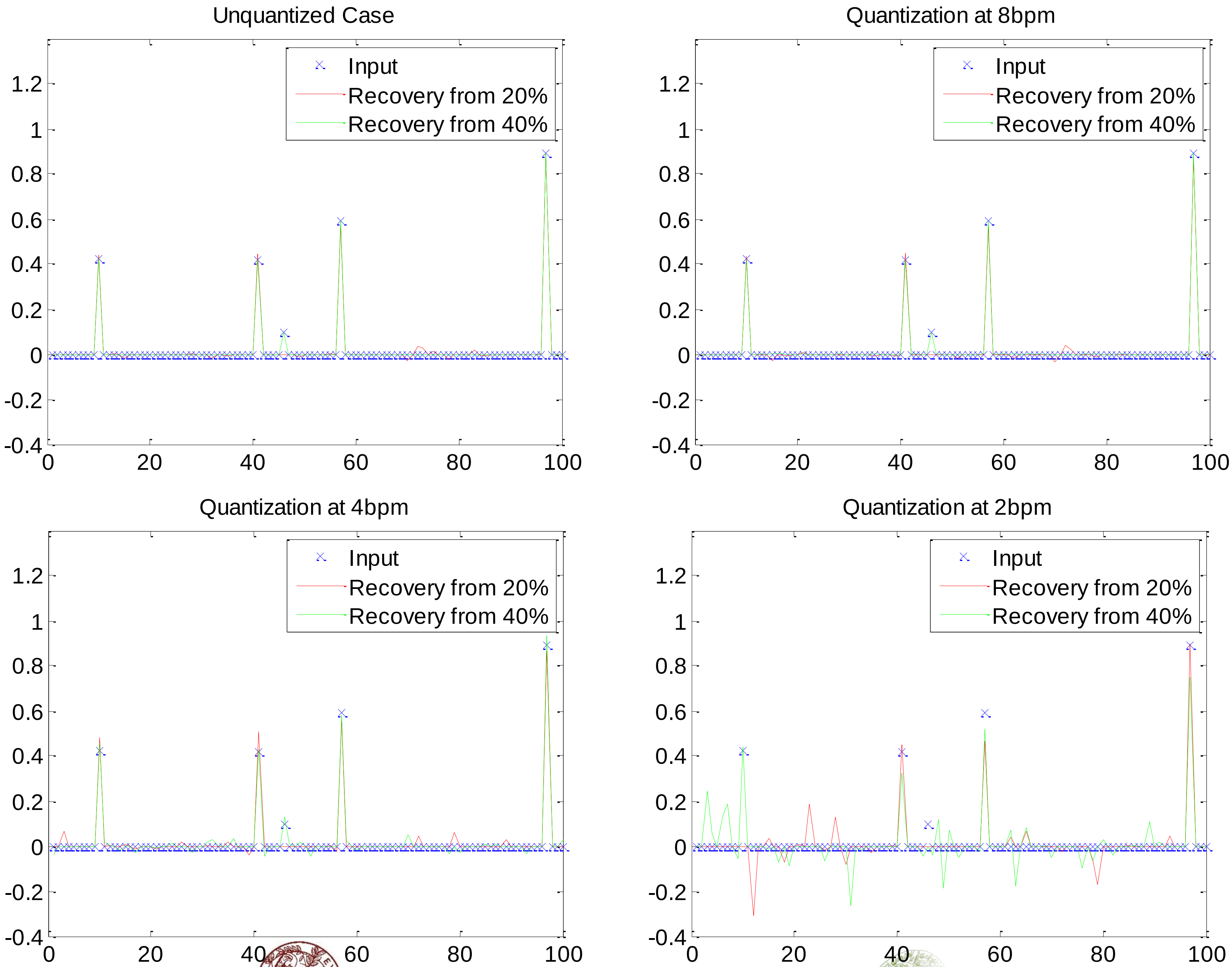
- Codebook $|\mathcal{C}| = \mathcal{B} = 2^R$
- Thresholds $\mathcal{T} = \{t_1, ..t_{\mathcal{B}} | t_i < t_j, \forall i, j\}$
- Dynamic range $M = |\max(x) - \min(x)|$
- Saturation $\hat{f} = \min(f, M)$

Uniform scalar

Non-uniform scalar: Lloyd-Max conditions

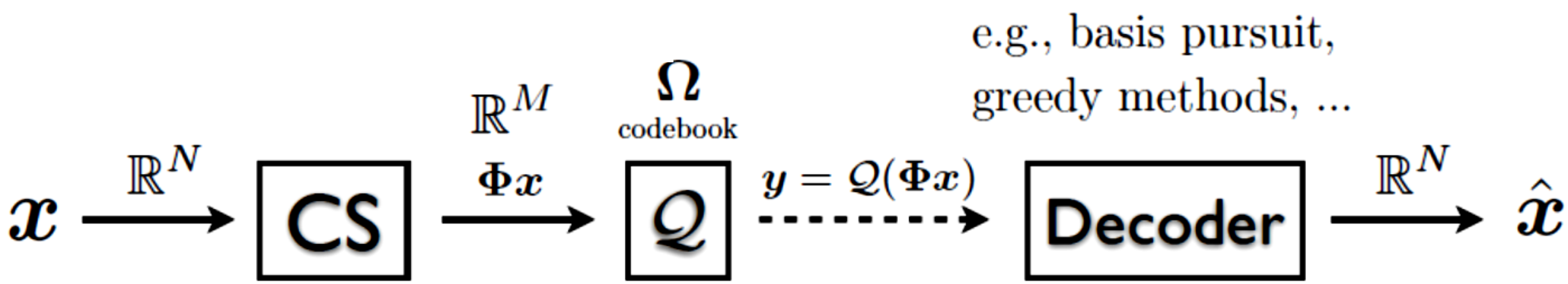
$$|t_j - t_i| = \Delta \qquad n = \log_2 M$$

$$t_j = \frac{1}{2}(\mathbb{E}\{X|X \in C_j\} + \mathbb{E}\{X|X \in C_{j+1}\})$$


Unquantized Case
Quantization at 8bpm
Quantization at 4bpm
Quantization at 2bpm
Input
Recovery from 20%
Recovery from 40%

Quantizing Compressed Sensing

- In a perfect noiseless system



Finite codebook $\Rightarrow \hat{\boldsymbol{x}} \neq \boldsymbol{x}$

i.e., impossibility to encode continuous domain
in a finite number of elements.

Objective: Minimize $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$

given a certain number of:
bits, measurements, or bits/meas.

Addressing Quantization in CS

- Operating regimes
  - Measurement Compression
  - Quantization Compression

➢ High resolution model (noise source)

$$\Delta << \|\mathbf{s}\|_2 \longrightarrow \Delta \approx \epsilon_Q : \min \|\mathbf{s}\|_1 \;\; \text{s.t.} \;\; \|\mathbf{y} - \mathbf{\Phi s}\|_2 \leq \epsilon_Q$$

➢ 1-bit CS $\mathbf{y} = \text{sign}(\mathbf{\Phi s})$

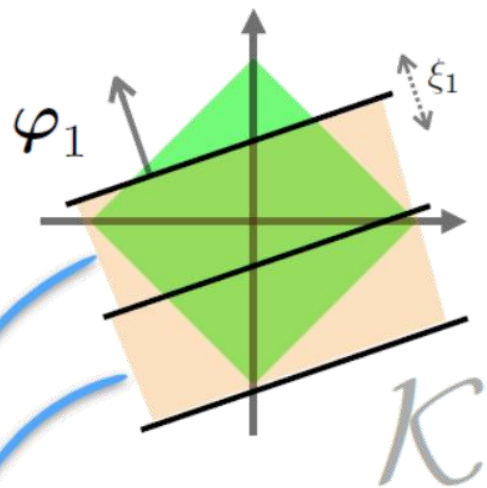
J. Laska, P. Boufounos, M. Davenport, and R. Baraniuk, “Democracy in action: Quantization, saturation, and compressive sensing," Applied and Computational Harmonic Analysis , 2011.

J.N. Laska, and R.G. Baraniuk. "Regime change: Bit-depth versus measurement-rate in compressive sensing." *Signal Processing, IEEE Transactions on Signal Processing* , 2012.

L. Jacques, J.N. Laska, P.T. Boufounos, and R.G. Baraniuk, “Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors," IEEE Transactions on Information Theory, 2013.

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 1. For consistent reconstruction method



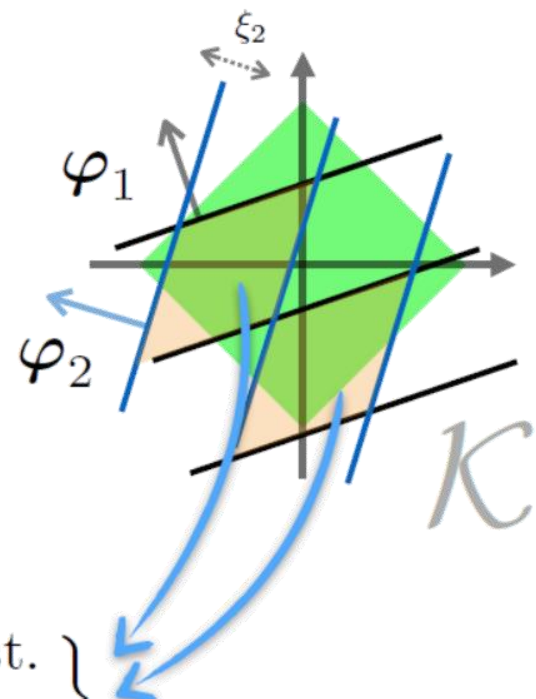
$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\underbrace{\mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)}_{\delta\lfloor(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1)/\delta\rfloor} = \text{cst.}$$

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})$

- 1. For consistent reconstruction method



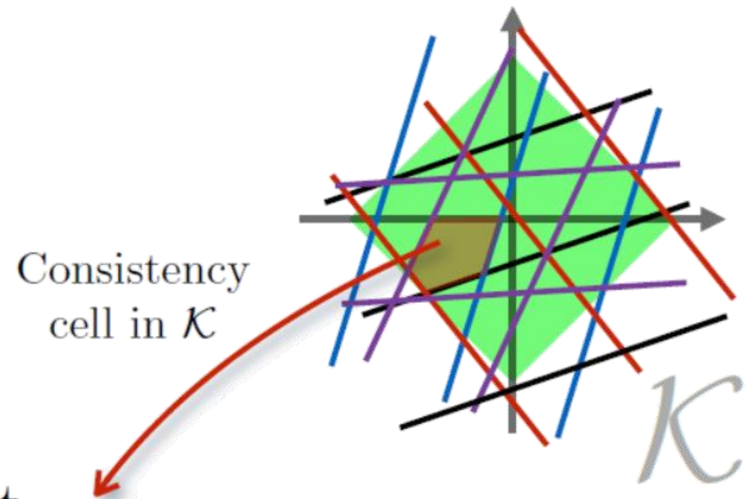
$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Signals $\boldsymbol{u}$ s.t.

$$\left.\begin{aligned} \mathcal{Q}(\boldsymbol{\varphi}_1^\top \boldsymbol{u} + \xi_1) &= \text{cst.} \\ \mathcal{Q}(\boldsymbol{\varphi}_2^\top \boldsymbol{u} + \xi_2) &= \text{cst.} \end{aligned}\right\}$$

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 1. For consistent reconstruction method



$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

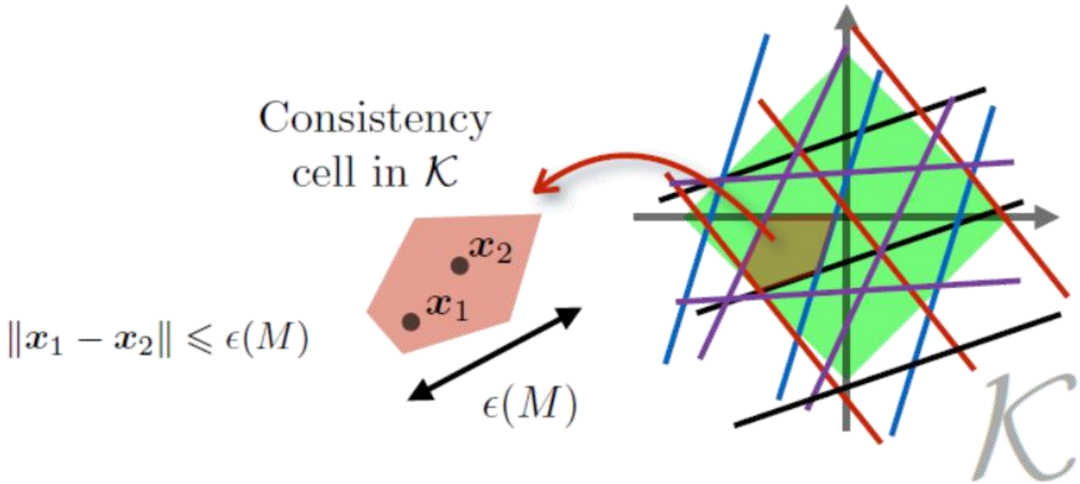
Signals $\boldsymbol{u}$ s.t.

$$\mathsf{A}(\boldsymbol{u}) := \mathcal{Q}(\boldsymbol{\Phi u} + \boldsymbol{\xi}) = \boldsymbol{y}$$

for some $\boldsymbol{y}$

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi x} + \boldsymbol{\xi})$

- 1. For consistent reconstruction method

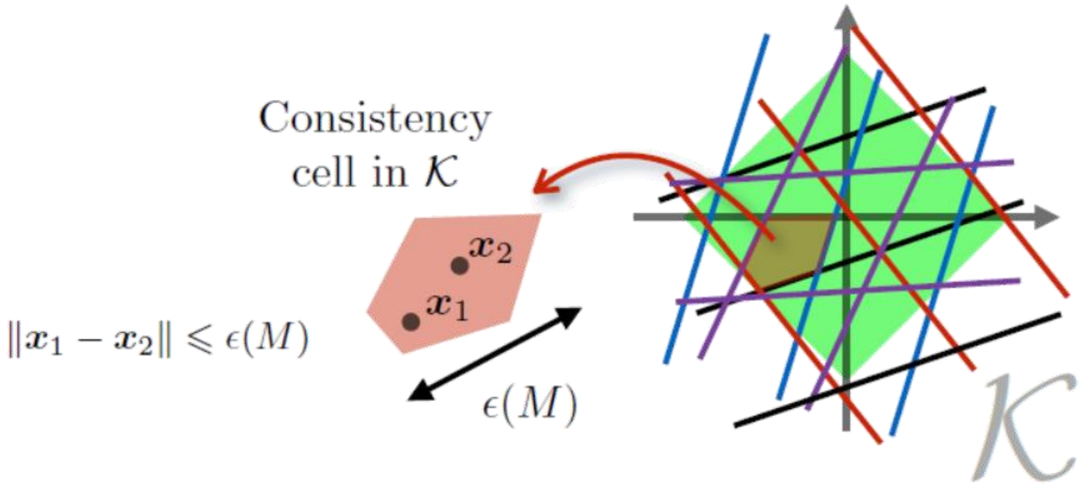


$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Size $\epsilon(M)$ should decay for large M!

Properties of $\mathsf{A}(\boldsymbol{x}) := \mathcal{Q}(\boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\xi})$

- 1. For consistent reconstruction method



$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\varphi}_1^T \\ \vdots \\ \boldsymbol{\varphi}_M^T \end{pmatrix}$$

Size $\epsilon(M)$ should decay for large M!

For $\boldsymbol{\Phi}$ a random Gaussian matrix, $\epsilon(M) \leqslant C_{\mathcal{K}}\, M^{-q}$.

with $q = 1$ if $\mathcal{K}$ is Σ_K, or low-rank matrices (and others),
and $q = \frac{1}{4}$ otherwise. **[LJ, 17]**

Proposed Recovery Mechanism

Introduce *quantization consistency*

Quantized Orthogonal Matching Pursuit (Q-OMP)

- Sparse signals

$$\min \|\mathbf{y} - \mathcal{Q}(\mathbf{\Phi s})\|_2^2 \ \text{ s.t. } \ \|\mathbf{s}\|_0 \leq K$$

- Compressible signals

$$\min \|\mathbf{y} - \mathcal{Q}(\mathbf{\Phi D x})\|_2^2 \ \text{ s.t. } \ \|\mathbf{x}\|_0 \leq K$$

Q-OMP

Algorithm 1: Quantized Orthogonal Matching Pursuit (Q-OMP)

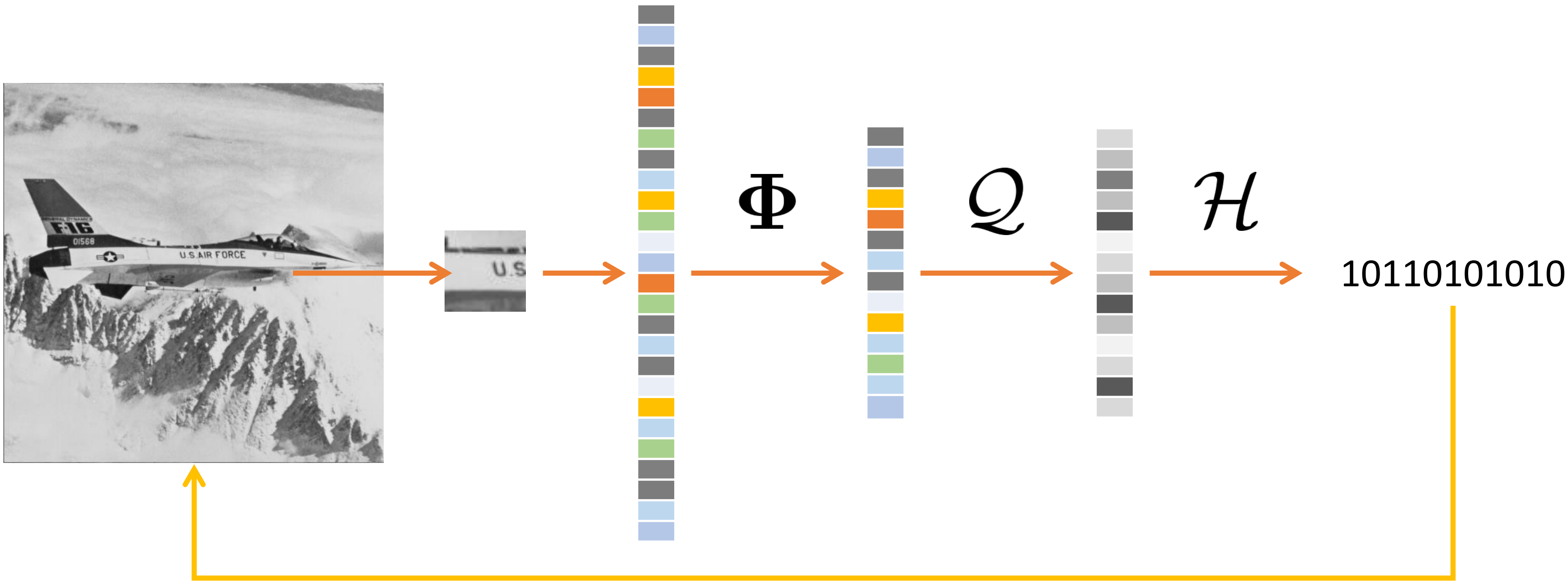
Input: The measurements $\mathbf{y}$,
The sensing matrix $\mathbf{\Phi}$,
The dictionary of examples $\mathbf{D}$,
The error tolerance *threshold* and/or maximum number of iterations k.

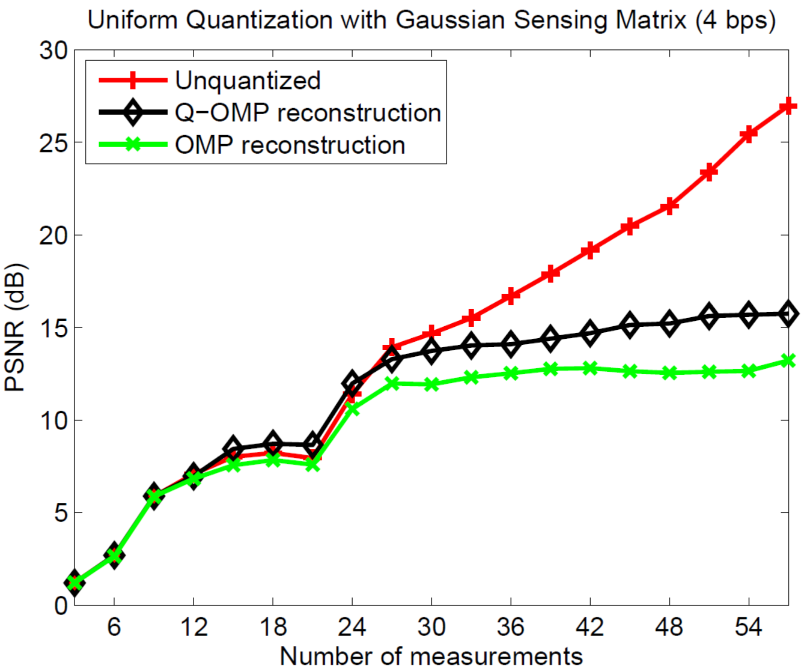
Output: The sparse representation coefficients $\hat{\mathbf{s}}$.

1: **initialization** $T^0 = \emptyset, \mathbf{r}^0 = \mathbf{y}$
2: **while** $error \geq threshold$ or $k \leq iteration\_lim$ **do**
3: $T^k = T^{k-1} \cup \arg\max_j |\mathcal{Q}(< \mathbf{r}^{k-1}, (\mathbf{\Phi D})_j >)|$.
4: $\hat{\mathbf{s}}_{T^k} = \arg\min_s \|y - \mathcal{Q}((\mathbf{\Phi D})_{T^k}\mathbf{s})\|_2$.
5: $\mathbf{r}^k = \mathbf{y} - (\mathbf{\Phi D})_{T^k}\hat{\mathbf{s}}_{T^k}$.
6: **set** $k \longleftarrow k+1$
7: **end while**

Experimental Setup

Sensing Matrices	Quantization
✓ Gaussian	✓ Uniform
✓ Binary	✓ Optimal




Uniform Quantization with Gaussian Sensing Matrix (4 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements
0
5
10
15
20
25
30
6
12
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24
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48
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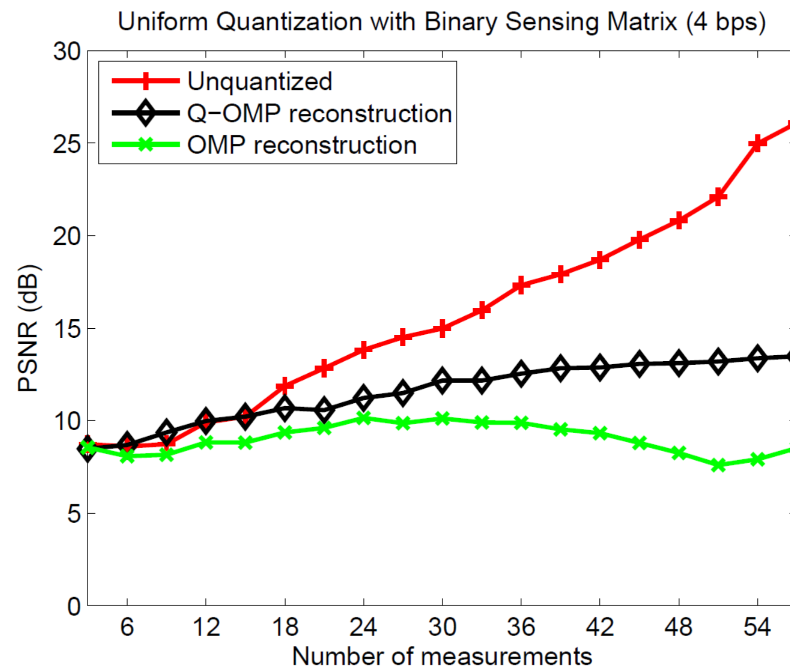
Uniform Quantization with Binary Sensing Matrix (4 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements
0
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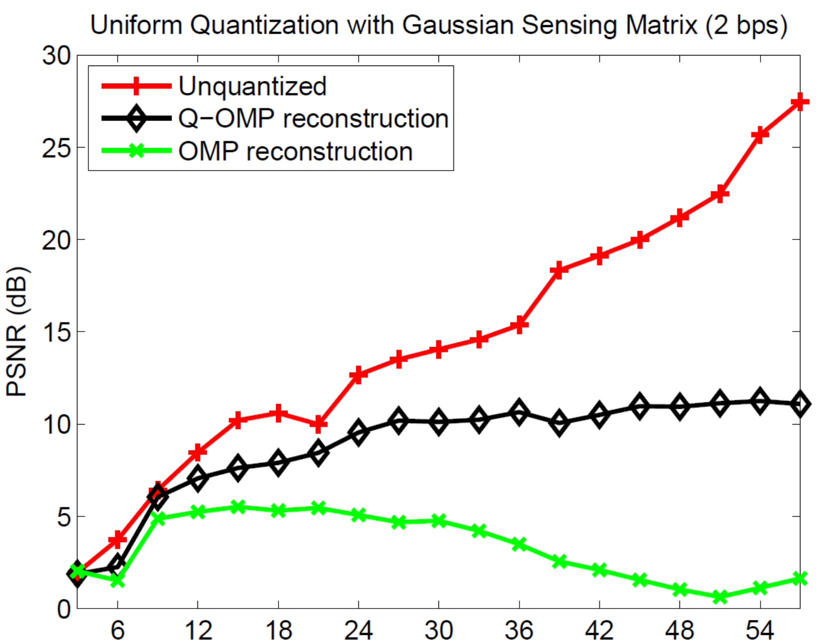
Uniform Quantization with Gaussian Sensing Matrix (2 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements
0
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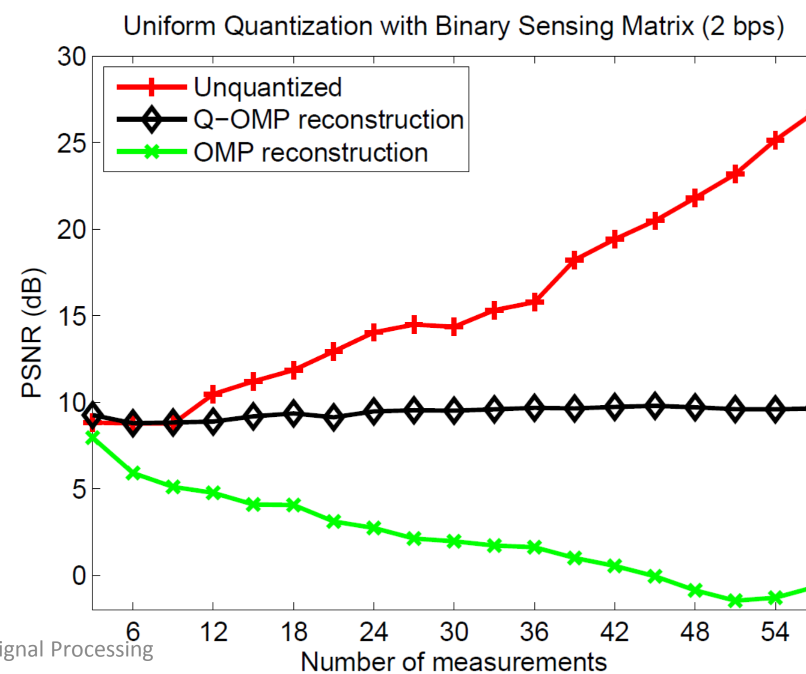
Uniform Quantization with Binary Sensing Matrix (2 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements
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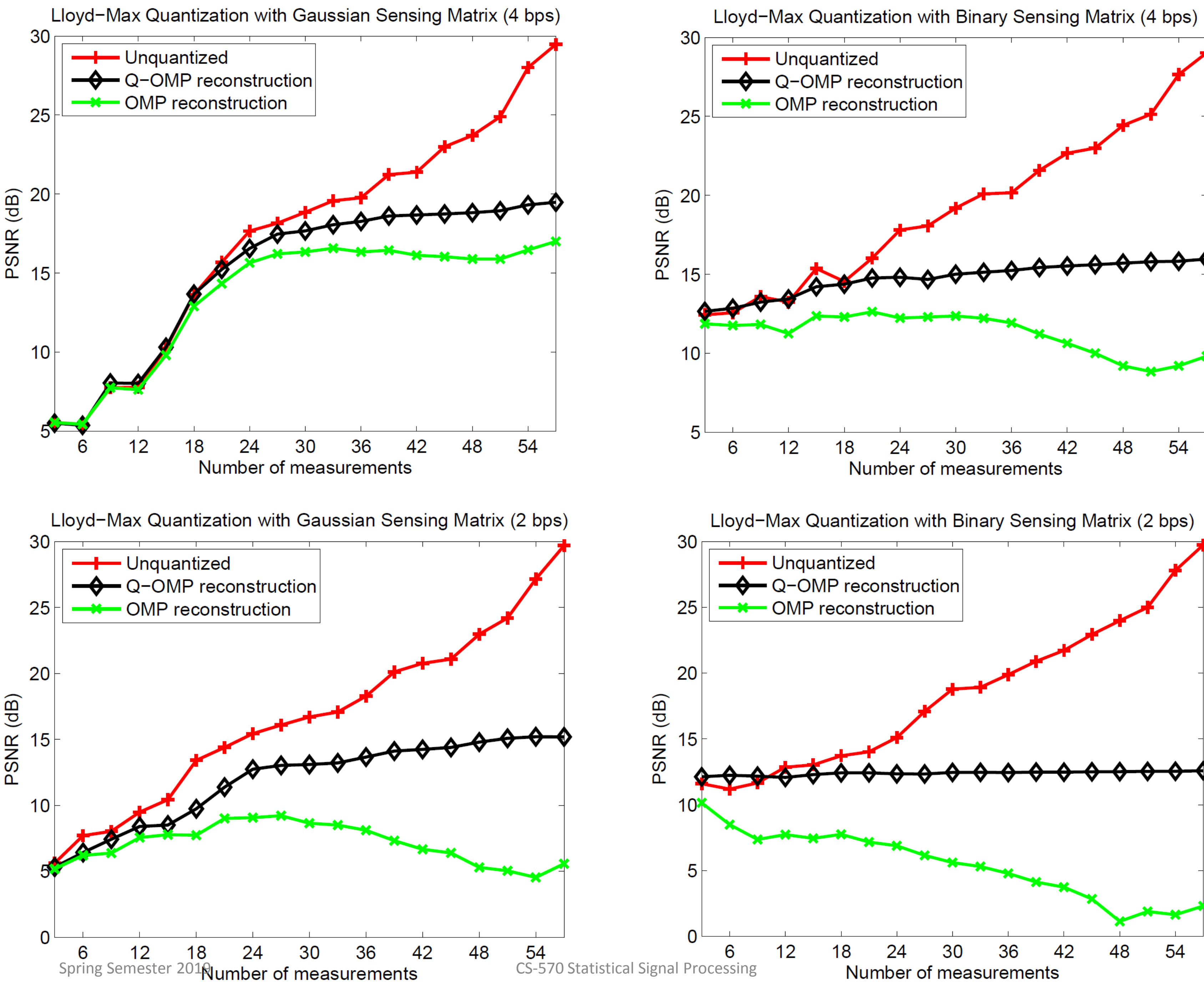
Lloyd−Max Quantization with Gaussian Sensing Matrix (4 bps)
Lloyd−Max Quantization with Binary Sensing Matrix (4 bps)
Lloyd−Max Quantization with Gaussian Sensing Matrix (2 bps)
Lloyd−Max Quantization with Binary Sensing Matrix (2 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements

Discussion

4 BPM		OMP	Q-OMP
Uniform	Gaussian	✓	✓
	Binary	✗	✓
Optimal	Gaussian	✓	✓
	Binary	✗	~

2 BPM		OMP	Q-OMP
Uniform	Gaussian	✗	✓
	Binary	✗	✓
Optimal	Gaussian	✗	✓
	Binary	✗	~

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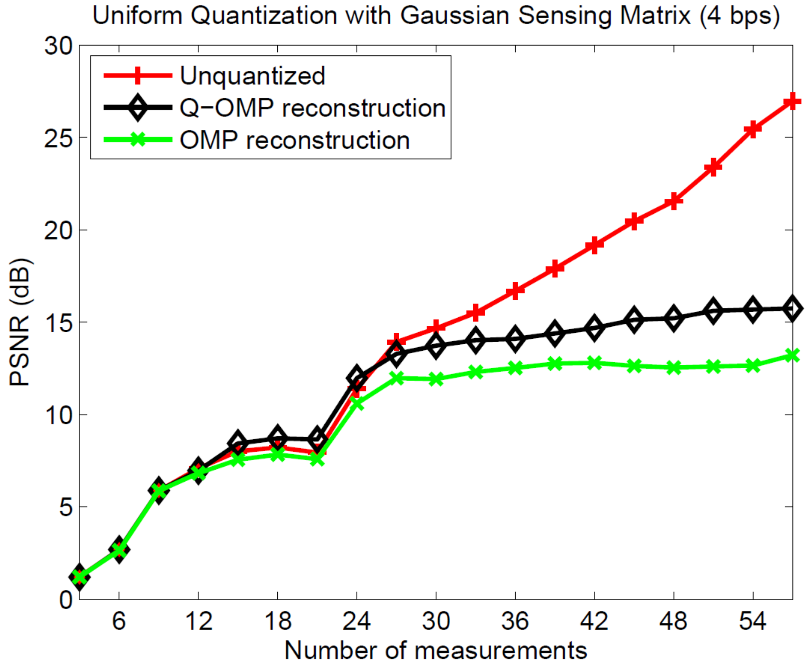
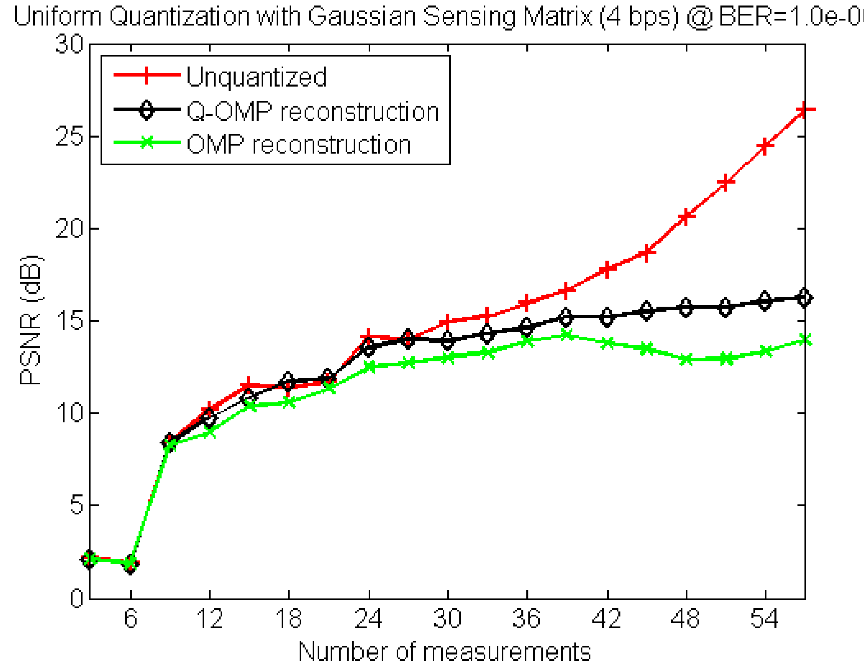
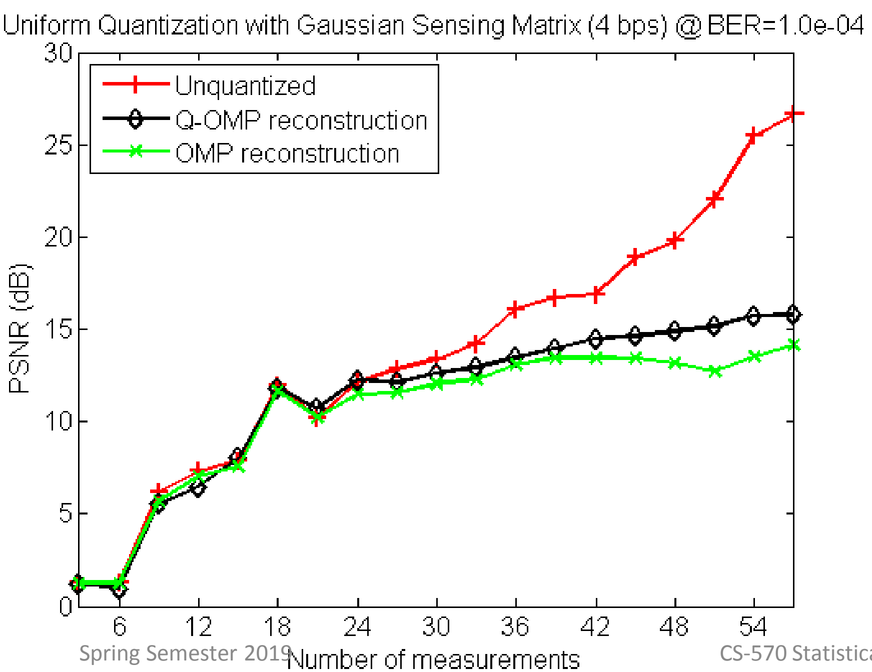
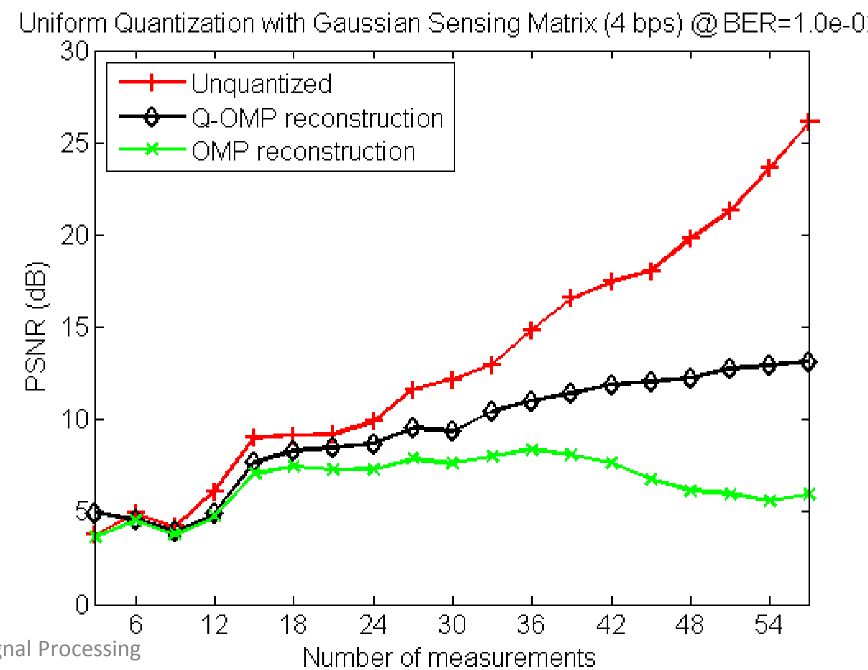
Source coding

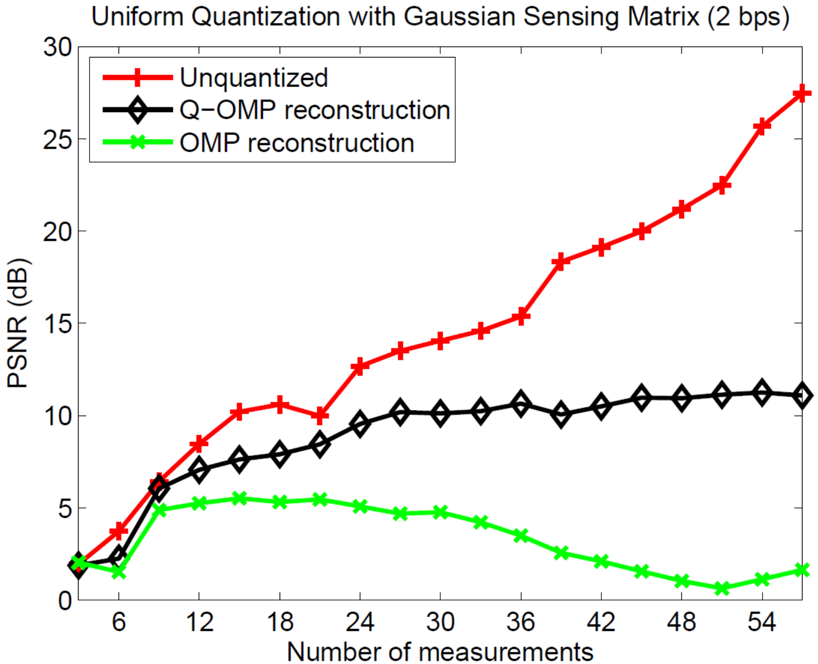
- Examine performance under realistic conditions
- Lossless coding: $\mathcal{H}$
  - Convert indices to binary vectors
  - Entropy coding
    - Huffman coding(codewords, prob.) -> binary code (JPEG)
    - Arithmetic coding (message) - > binary (JPEG2000)
  - Introduce bit errors
    - Storage
    - Transmission


Uniform Quantization with Gaussian Sensing Matrix (4 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements


Uniform Quantization with Gaussian Sensing Matrix (4 bps) @ BER=1.0e-06
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements


Uniform Quantization with Gaussian Sensing Matrix (4 bps) @ BER=1.0e-04
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements


Uniform Quantization with Gaussian Sensing Matrix (4 bps) @ BER=1.0e-02
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements


Uniform Quantization with Gaussian Sensing Matrix (2 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements
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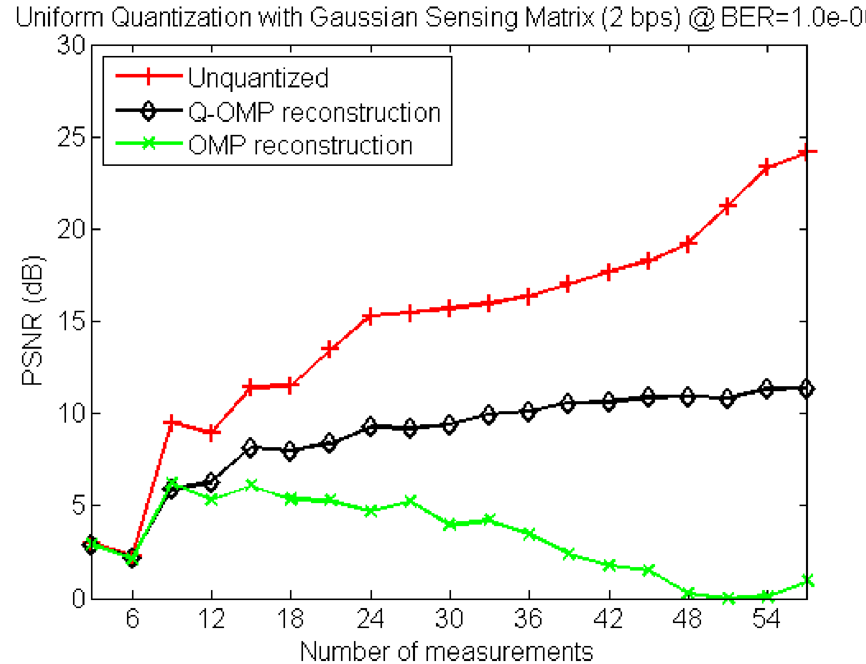
Uniform Quantization with Gaussian Sensing Matrix (2 bps) @ BER=1.0e-06
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
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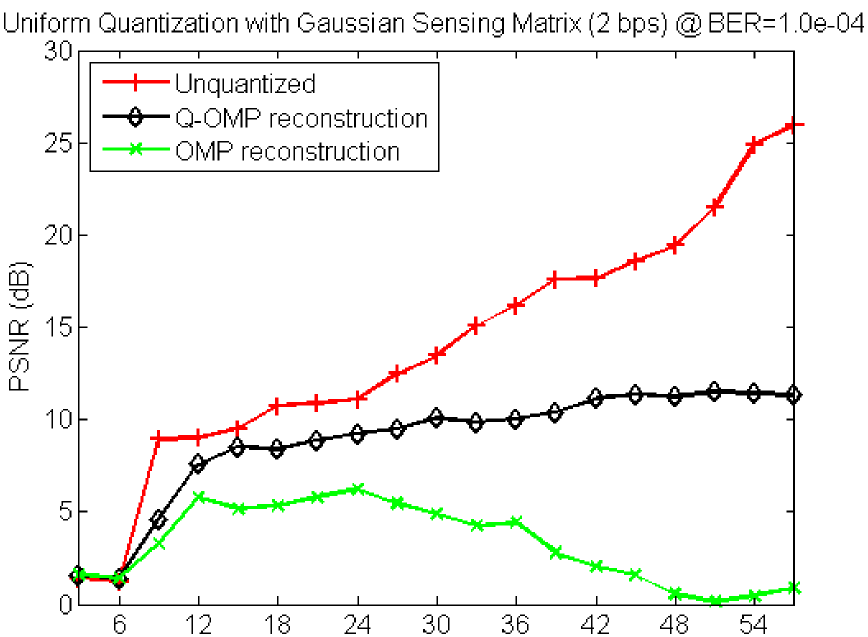
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Q-OMP reconstruction
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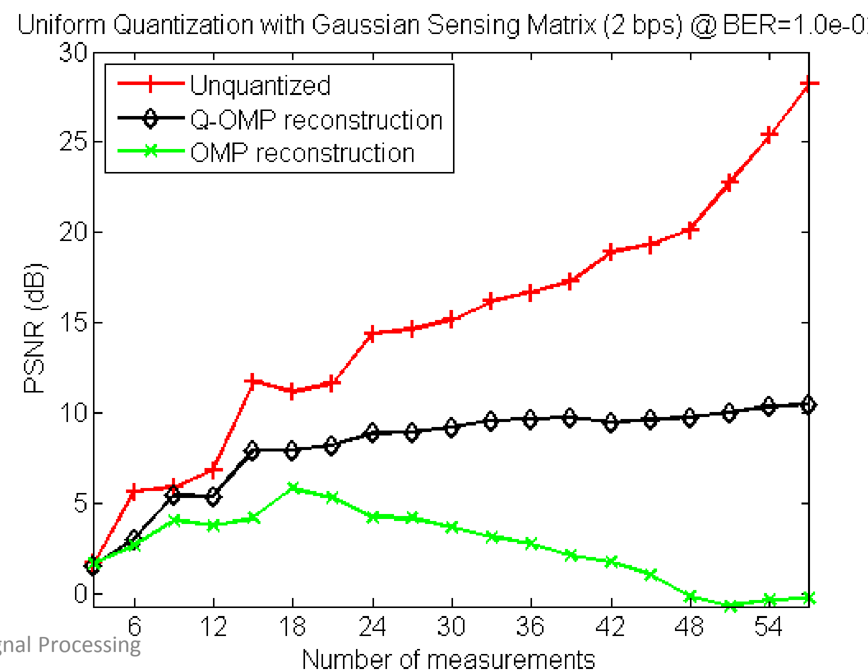
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Q-OMP reconstruction
OMP reconstruction
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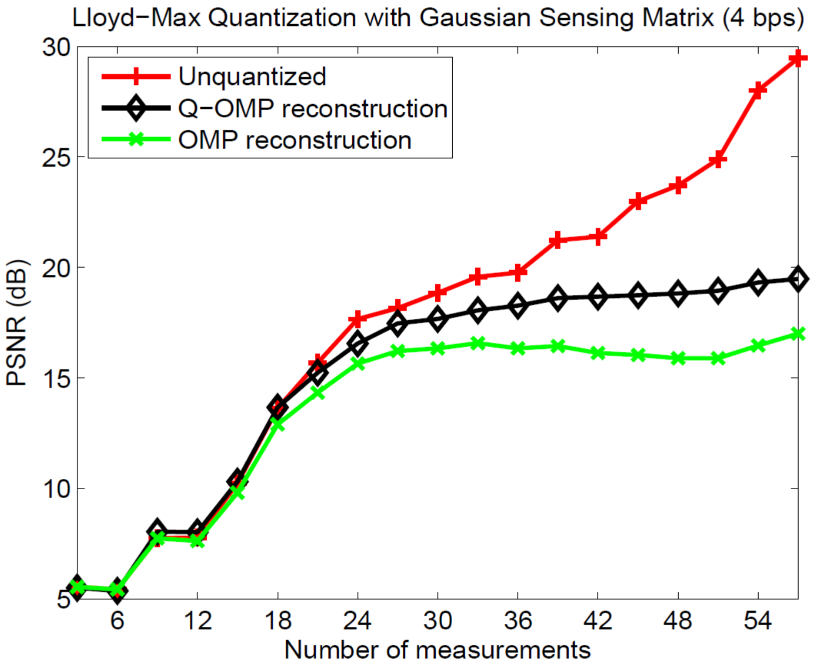
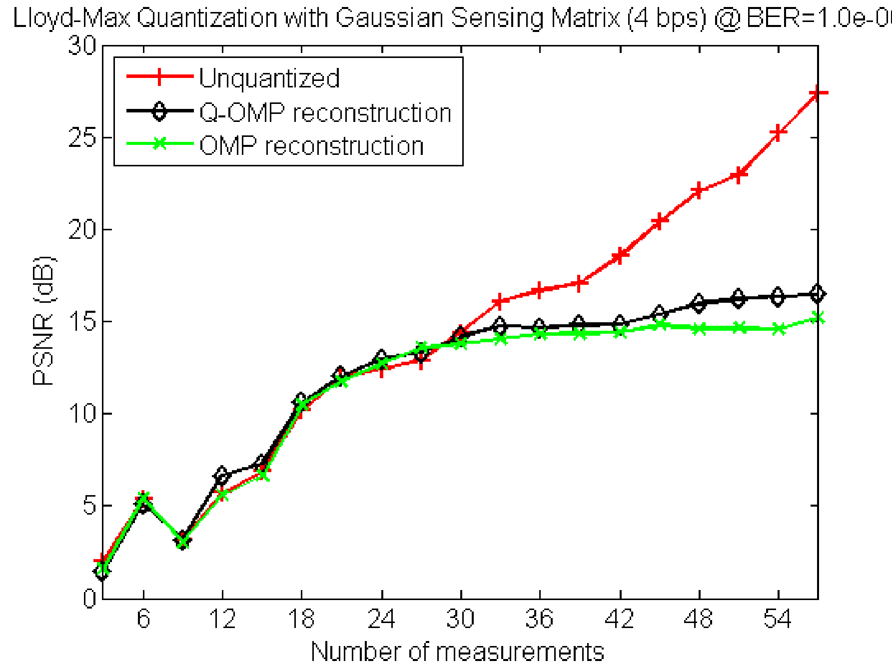

Lloyd−Max Quantization with Gaussian Sensing Matrix (4 bps)
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements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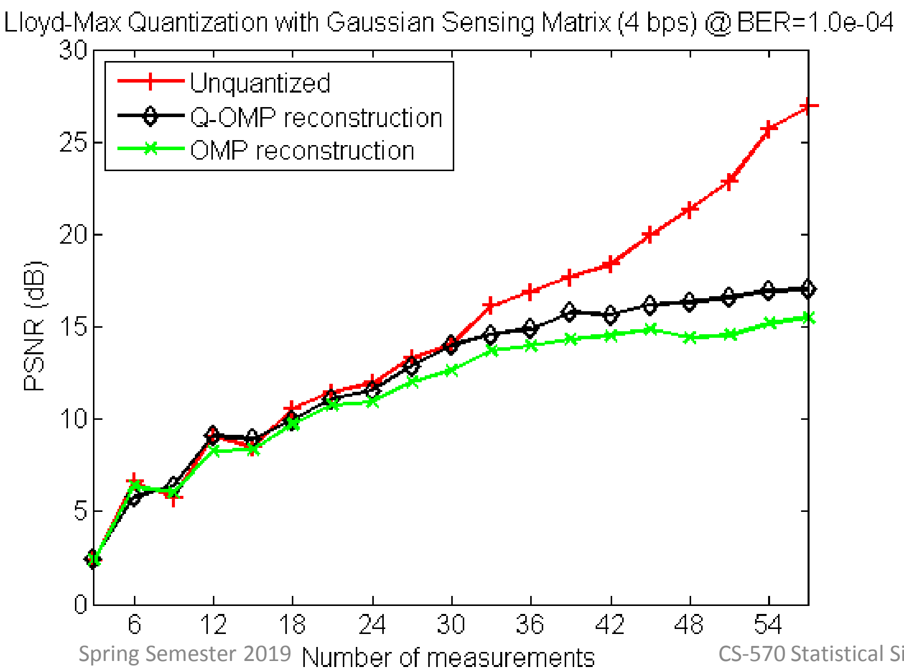
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements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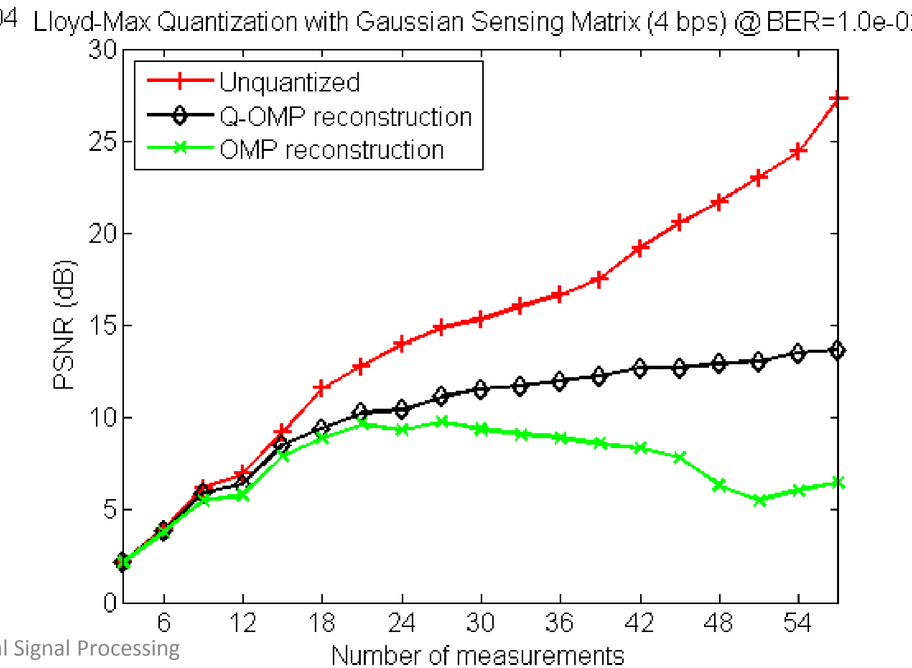
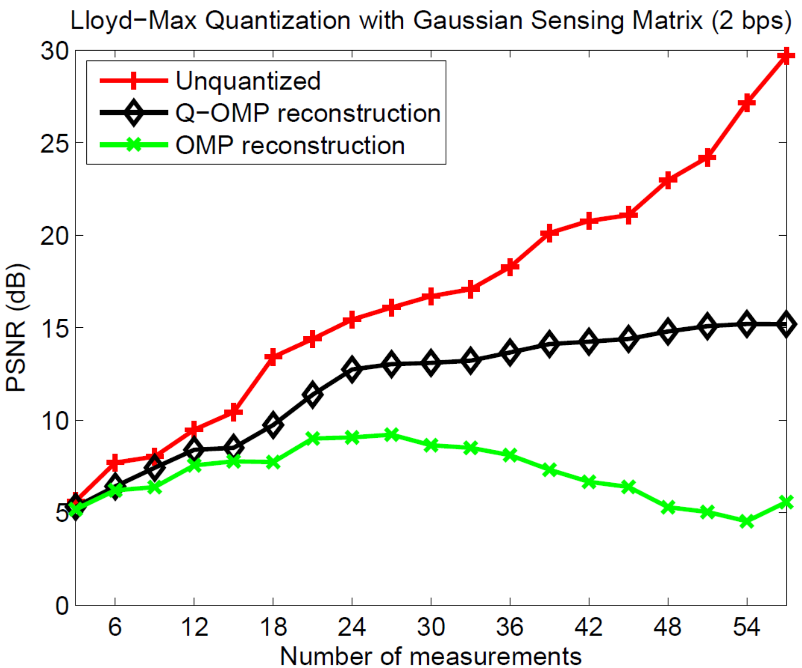
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements


Lloyd-Max Quantization with Gaussian Sensing Matrix (4 bps) @ BER=1.0e-02
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements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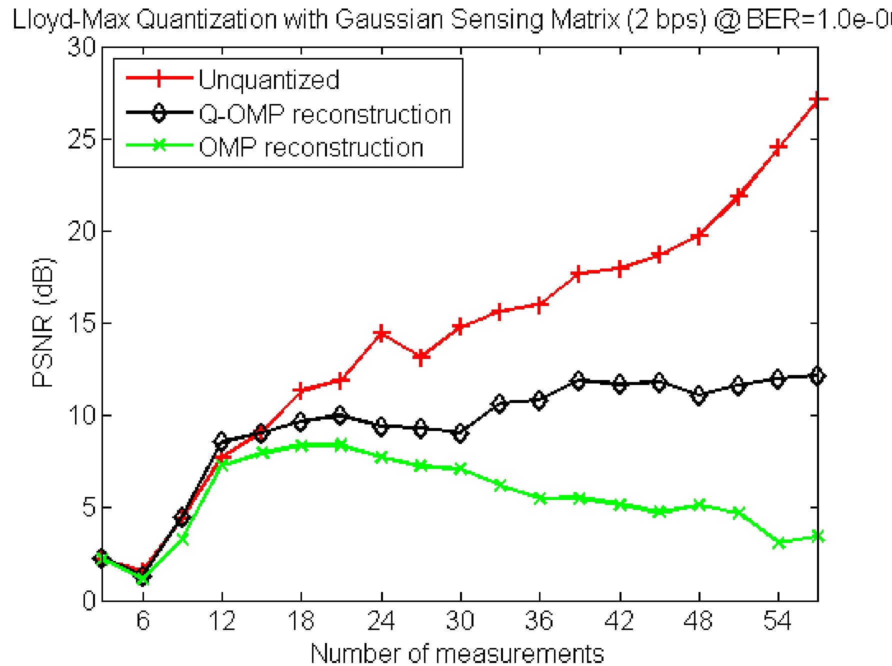
Unquantized
Q−OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements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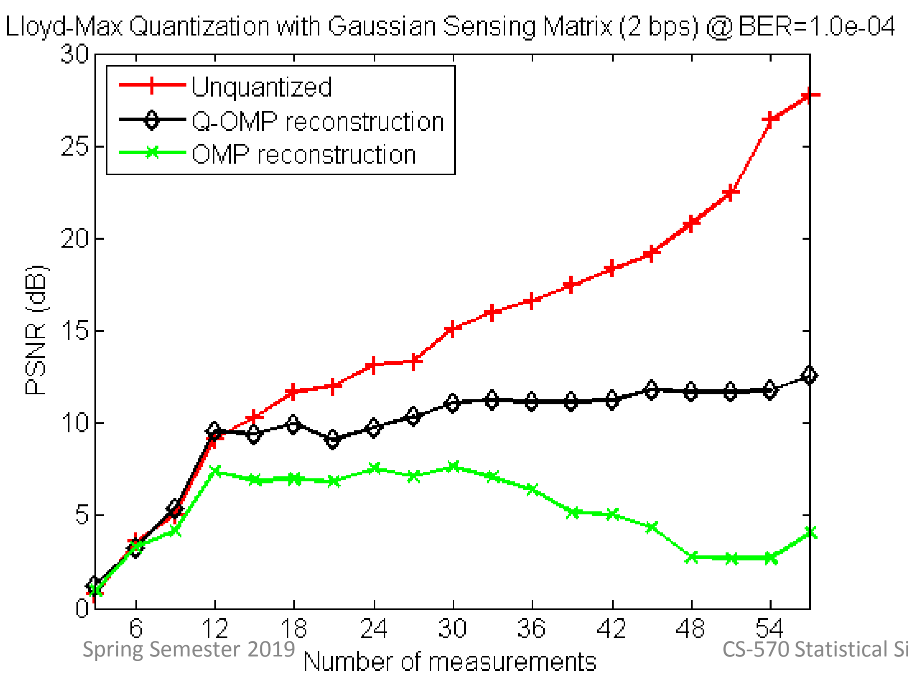
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements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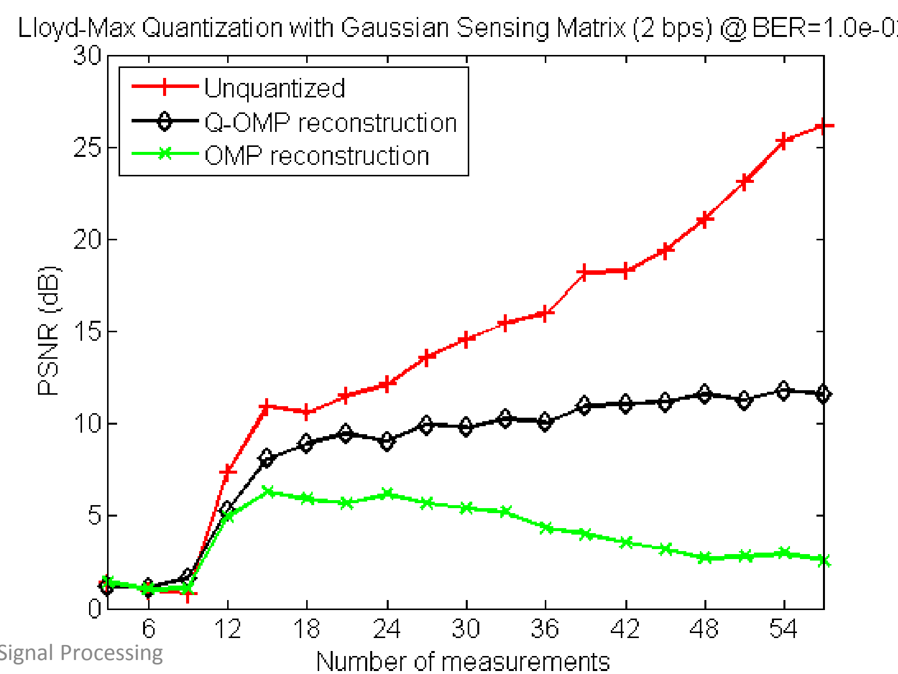

Lloyd-Max Quantization with Gaussian Sensing Matrix (2 bps) @ BER=1.0e-02
Unquantized
Q-OMP reconstruction
OMP reconstruction
PSNR (dB)
Number of measurements

Discussion

4 b.p.m		OMP	Q-OMP
Small	Uniform	✓	✓
	Optimal	✓	✓
Medium	Uniform	✓	✓
	Optimal	✓	✓
Large	Uniform	✗	✓
	Optimal	✗	✓

2 b.p.m		OMP	Q-OMP
Small	Uniform	✓	✓
	Optimal	✓	✓
Medium	Uniform	✓	✓
	Optimal	✓	✓
Large	Uniform	✗	✓
	Optimal	✓	✓

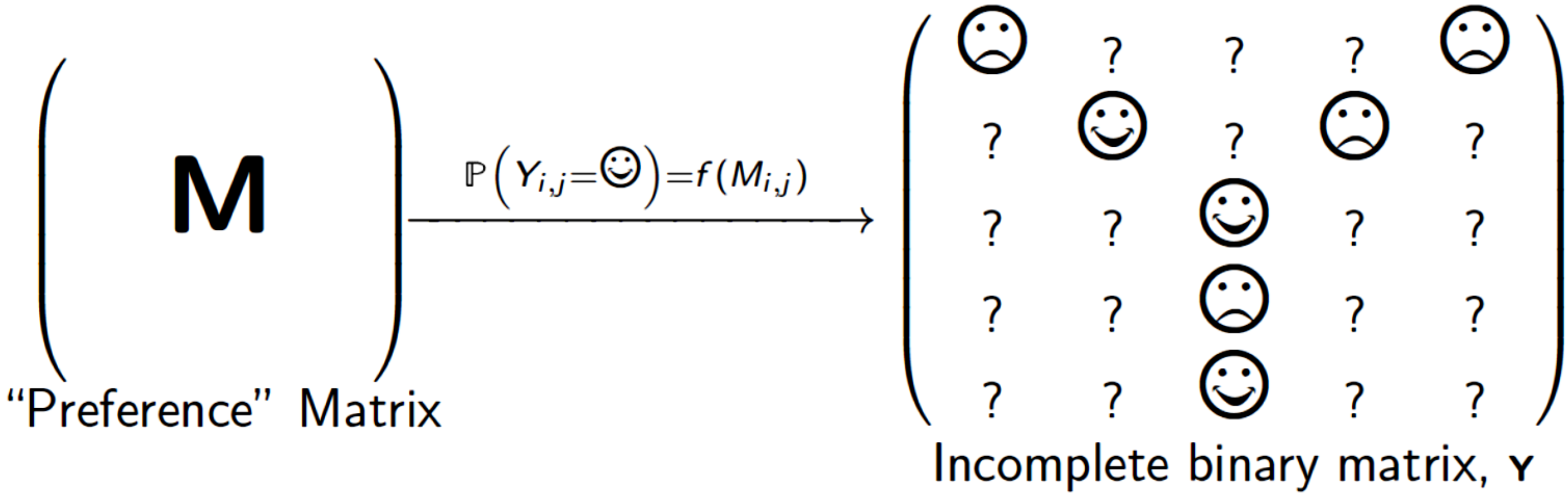
- Quantization is critical in CS
  - Little effect w.r.t uniform vs. scalar
  - Large effects on b.p.m & sampling scheme
  - Sensitive to BER
- Q-OMP
  - Large effect w.r.t optimality
  - Smaller effect on b.p.m & sampling scheme
  - Robust to BER

1-bit Matrix Completion

- Netflix challenge v2

	John	Anne	Scot	Mark	Alice
Chicago	☺	☹	?	?	?
Matrix	☹	?	☺	?	?
Star wars	?	?	☺	?	☹
Inception	?	☺	?	☹	?
Alien	☺	☹	?	?	?
Pulp Fiction	?	?	☹	?	☺

Generalized linear model



- $\mathbf{M}$ is unknown. $\mathbf{M}$ has (approximately) low rank.
- $f : \mathbb{R} \to [0,1]$ is a known function (e.g., the logistic curve).
- $\mathbf{M} \in \mathbb{R}^{d \times d}, \mathbf{Y} \in \{☺, ☹\}^{d \times d}$.
- $\Omega \subset \{1, 2, \ldots, d\} \times \{1, 2, \ldots, d\}$. You see $\mathbf{Y}_\Omega$.

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Formulation

- Consider a $d \times d$ matrix **M** with rank *r*. Suppose we observe a subset Ω of entries of a matrix **Y** in the following way:

$$Y_{i,j} = \begin{cases} +1 & \text{if } M_{i,j} + Z_{i,j} \geq 0 \\ -1 & \text{if } M_{i,j} + Z_{i,j} < 0 \end{cases}$$

Observations Matrix Noise

- Generalized model: M is $d_1 x d_2$, and $f : \mathbf{R} \to [0,1]$

$$Y_{i,j} = \begin{cases} +1 & \text{with probability } f(M_{i,j}), \\ -1 & \text{with probability } 1 - f(M_{i,j}) \end{cases} \quad \text{for } (i,j) \in \Omega.$$

Equivalence when $f(x) = \frac{e^x}{1+e^x}$ and Z i.i.d logistic distribution

Estimation

- Log-likelihood function

$$\mathcal{L}_{\Omega,\mathbf{Y}}(\boldsymbol{X}) := \sum_{(i,j)\in\Omega} \left(\mathbb{1}_{[Y_{i,j}=1]} \log(f(X_{i,j})) + \mathbb{1}_{[Y_{i,j}=-1]} \log(1 - f(X_{i,j})) \right)$$

- Recovery of **M**

$$\widehat{\boldsymbol{M}} = \arg\max_{\boldsymbol{X}} \mathcal{L}_{\Omega,\mathbf{Y}}(\boldsymbol{X}) \quad \text{subject to} \quad \|\boldsymbol{X}\|_* \le \alpha\sqrt{r d_1 d_2} \quad \text{and} \quad \|\boldsymbol{X}\|_\infty \le \alpha.$$

- Error

Let f be the logistic function. Assume that $\frac{1}{d}\|\mathbf{M}\|_* \le \sqrt{r}$. Suppose the sampling set is chosen at random with $\mathbb{E}|\Omega| = m \ge d\log(d)$. Then with high probability,

$$\frac{1}{d^2}\sum_{i,j} d_H^2(f(\hat{M}_{i,j}), f(M_{i,j}))^2 \le C \min\left(\sqrt{\frac{rd}{m}}, 1\right).$$